SECONDARY-SCHOOL MATHEMATICS

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BOOK I

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PREFACE

THIS text differs widely from that marked out by custom and tradition. It treats the various branches of mathematics more with reference to their unities and less as isolated entities (sciences). It seeks to give pupils usable knowledge of the principles underlying mathematics and ready control of them. These texts are not an experiment; they were thoroughly tried out in mimeograph form on hundreds of high school pupils before being put into book form.

The scope of Books I and II does not vary greatly from that covered in algebras and geometries of the usual type. However, Book I is different in that arithmetic, algebra, and geometry are treated side by side. The effect of this arrangement is increased interest and power of analysis on the part of the learner, and greater accuracy in results. Some pupils like arithmetic, others like algebra, still others like geometry; the change is helpful in keeping up interest. The study of geometry forces analysis at every step and stage; consequently written problems and problems to be stated have no terrors for those who are taught in this way.

For several years mathematical associations have urged that all work should be based upon the equation. In accordance with this view we have made the demonstrations in this book largely algebraic, thus making the demonstration essentially a study in simultaneous equations.

In this method of treatment, we have found it advantageous not to hurry the work. Pupils gain power, not in solving many problems, but in analyzing and thoroughly *understanding the principles* of a few.

Book I covers straight line geometry to proportion, and algebra through fractional equations; it is intended for one year's work.

Book II completes straight line geometry, takes up the study of the circle, and on the algebraic side takes the student through simultaneous quadratics.

We are indebted to many who have offered suggestions and practical problems, and especially to Carlotta Greer of Cleveland Technical High School, and Professor Kenneth G. Smith of the University of Wisconsin; also to those who so kindly read the proof sheets. We invite criticisms and suggestions from teachers who are interested in progressive work along these lines.

FOR THE TEACHER

Reviews in mathematics are always necessary. This is especially true in this text, which combines different branches of mathematics.

In teaching the text keep in mind that in geometry as well as in algebra problems are solved by means of equations. The equation is the principal tool used. To use the equation method successfully, the Hilbert notation, a small letter for angle values and for line values is essential.

In lettering a figure, begin at the lower left-hand corner and read counter-clockwise. This gives pupils an idea of directed lines, and makes possible the correct drawing of the figure from description.

Emphasize the manipulation of quantities by means of factors and the use of methods of indication until there is no longer hope that the factors may disappear through division (see p. 129).

Do a large amount of the work orally, and do it so often that the pupil *knows* what he is doing, and why he is doing it. No pupil should use pencil and paper to find prime factors of $(24)^2$, $(12)^6$, $9 \cdot 27$, or to find the product of $18 \cdot 17$.

Note that many demonstrations have been put into the form of a set of simultaneous equations, the solution of which produces the desired equation.

Insist that pupils study all illustrative work, rules, and instructions before beginning the examples of an exercise.

Teach pupils to use the Index, also the groups of theorems and constructions found on pages 175–179.

Book I is considered one year's work. Book II is for the *second year.*

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SECONDARY-SCHOOL MATHEMATICS

BOOK I

CHAPTER I

The Number System

1. Our number system is a decimal one. Ten units of one order make one of the next higher. One tenth of any digit makes a digit of the next lower order. The digits are 1, 2, 3, 4, 5, 6, 7, 8, 9. All numbers are made up of these digits and their position is often indicated by the introduction of one other symbol, 0, known as nought, cipher, or *zero; e.g.*, 16 is equal to ten 1's added to six 1's. That is, $16 = 10 + 6$, the position of the 1 being indicated by the 0.

The numbers expressed by these digits themselves are often the multiples of other digits. For example, $4 =$ two 2's, or $2 \cdot 2$, where the $\cdot$ indicates multiplication.

$$6 = 3 \cdot 2, \qquad 9 = 3 \cdot 3.$$

The numbers 1, 2, 3, 5, 7 are the **prime digits.** A **prime number** is a number whose only integral factors are itself and unity.

EXERCISE 1

Write the following numbers in such a way that their decimal composition will appear:

1. 145.

$145 = 100 + 40 + 5.$

2. 223. **3.** *448.* **4.** 987. **5.** 999. **6.** 227. **7.** 863. **8.** 511. **9.** 101. **10.** 10016.

2. In the product of two or more numbers, any one of them or the product of any number of them is a **factor** of the *given product.* $2 \cdot 3 \cdot 5 = 30$. Then, 2 is a factor of 30. $2 \cdot 3$ or 6 is also a factor of 30.

3. A **term** is a number whose parts are not separated by the plus (+) or minus (−) sign.

In the expression,

$$10 + 6$$

10 is a term. 6 is also a term.
$10 + 6$ is composed of two terms.

4. A **binomial** is an expression of *two* terms.

A **trinomial** is an expression of *three* terms.

A **quadrinomial** is an expression of *four* terms.

An expression of two or more terms is also called a **polynomial.**

$100 + 60 + 3$ is a trinomial.

5. It is often necessary to represent a number by a letter or a combination of letters. Such letters may represent either unknown numbers or those *supposed* to be *known* numbers. This kind of notation is used in *general arithmetic* or *algebra.*

E.g., n may represent any number, likewise any letter or combination of letters and figures may be considered a number.

$a + b + c$ is a trinomial number (§ 4), or the sum of three numbers a, b, and c. In arithmetic it is possible to express such a sum as a single number.

Thus, $2 + 5 + 8 = 15$.

In algebra, this is not possible unless the terms of the expression are alike or similar.

***6. Similar* Terms** are terms which differ in their coefficients *only, e.g.,* $5 \cdot x$, $6 \cdot x$, $a \cdot x$, $b \cdot x$.

7. Any factor (§ 2) of a number is the **coefficient** of the remaining factors.

Thus, in $a \cdot x$, a is the coefficient of x.
in $2 \cdot 3$, 2 is the coefficient of 3.
in $2 \cdot 3$, 3 is the coefficient of 2.
in $2 \cdot a \cdot b$, $2 \cdot b$ is the coefficient of a.
in $2 \cdot a \cdot b$, a is the coefficient of $2 \cdot b$.
in $2 \cdot a \cdot b$, 2 is the *numerical* coefficient of $a \cdot b$.
in ax, a is the *literal* coefficient of x.

When the product of a number of figures and letters is to be written, the multiplication sign is usually omitted.

Thus, $2 \cdot a \cdot b$ is written $2\,ab$.

8. $2\,a + 3\,a + 7\,a$ is a trinomial consisting of similar terms (§ 6). These terms may be united into one term by finding the sum of the coefficients.

Hence, $2\,a + 3\,a + 7\,a = (2 + 3 + 7)\,a = 12\,a$.

This is the same operation as that in arithmetic when one finds the value of

$$2 \text{ ft.} + 3 \text{ ft.} + 7 \text{ ft.} = 12 \text{ ft.},$$

and is brought still closer to arithmetic when one remembers that only like numbers can be added.

Similarly, $15\,ab - 3\,ab + 7\,ab$

means that $3\,ab$ *is to be subtracted from* $15\,ab$, *and* $7\,ab$ *added to this difference.*

Hence, $15\,ab - 3\,ab + 7\,ab = 19\,ab$.

Ex. 1. Find the sum of $20\,xy + 4\,xy - 7\,d$.

$$20\,xy + 4\,xy - 7\,d = (20\,xy + 4\,xy) - 7\,d = 24\,xy - 7\,d.$$

Ex. 2. Add $5\,a^2 + 3\,ab + 4\,b^2$ and $-4\,a^2 - 2\,ab - 4\,b^2$.

$$\begin{array}{r} 5\,a^2 + 3\,ab + 4\,b^2 \\ -4\,a^2 - 2\,ab - 4\,b^2 \\ \hline a^2 + ab \phantom{{}- 4\,b^2} \end{array}$$

EXERCISE 2

Find the sum of the following.

1. $21x + 9x - 4x + 3x - 8x$.

$$21x + 9x = 30x$$
$$30x - 4x = 26x$$
$$26x + 3x = 29x$$
$$29x - 8x = 21x.$$

This work is all to be done mentally, only results of each addition being given.

2. $5m - 4m + 6m - 2m$.

3. $8xy + 3xy - 2d$.

4. $8a + 4b$, $4a - 2b$.

5. $16a^2 + 8ab + 5b^2$, $5a^2 - 3ab + 2b^2$.

6. $21x^2 + 22xy + 17y^2$, $-8x^2 + 2xy - 9y^2$, and $7x^2 - 11xy - 7y^2$.

7. $24a^2 + 48ab + 24b^2$, $-23a^2 - 47ab + 23b^2$, and $a^2 + 2ab - b^2$.

8. $14c^2 + 21cd + 10d^2$, $-9c^2 - 12cd - d^2$, and $-5c^2 - 9cd - 9d^2$.

9. $3 \cdot 19 + 2 \cdot 19 + 5 \cdot 19$.

10. $3 \cdot 27 + 2 \cdot 27 - 4 \cdot 27$.

11. $14 \cdot 18 + 25 \cdot 18 - 16 \cdot 18 - 12 \cdot 18$.

12. $41 \cdot 63 - 27 \cdot 63 - 12 \cdot 63$.

13. Express 27 as a binomial.

14. If x is the digit in tens' place and y in units' place, express the number as a binomial.

15. Express 47 as a binomial.

16. Express 648 as a trinomial.

17. If hundreds' digit is x, tens' digit y, units' digit z, express the number as a trinomial.

18. If the digits of example 17 are *reversed*, express the number.

9. We have considered the *decimal phase* of our number system; the *prime factors* are of equal importance.

The prime factors of 15 are 3 and 5.

$45 = 3 \cdot 3 \cdot 5$ or $3^2 \cdot 5$, where the ^{2} indicates the number of times 3 occurs as a factor.

EXERCISE 3

1. *Learn* the following squares:
$1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, \cdots 30^2$.

2. *Learn* the following cubes:
$1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, 9^3, 10^3, 11^3, 12^3$.

10. Literal monomials may be separated into factors.

Thus,
$$a^3 = a \cdot a \cdot a.$$
$$a^2b = a \cdot a \cdot b.$$
$$(ab)(ab)(ab) = (ab)^3 = a^3b^3.$$

Similarly, $(2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (2 \cdot 3)^3 = 2^3 \cdot 3^3 = 6^3$.

Ex. Find the prime factors of 225.
$$225 = (15)^2 = (3 \cdot 5)^2 = 3^2 \cdot 5^2.$$

EXERCISE 4

Find the prime factors of the following:

1. 18, 27, 24.

2. $(18)^2, (22)^2$.

3. 361, 520.

4. $9^3, (27)^2$, 729.

5. $(12)^2, (12)^3$.

6. $(60)^2$.

7. 484.

8. $(36)^2$.

9. 625.

10. $225 \cdot 72$.

11. An expression that is a factor of each of two or mo: expressions is said to be a **common factor** of them.

Thus,
$$15\,a^2b = 3 \cdot 5 \cdot a \cdot a \cdot b.$$
$$25\,a^2b = 5^2 \cdot a \cdot a \cdot b.$$
$$10\,a^3c = 2 \cdot 5 \cdot a \cdot a \cdot a \cdot c.$$

5, a, and a are the factors common to each of the numbers $15\,a^2$ $25\,a^2b$, $10\,a^3c$.

EXERCISE 5

Find the factors common to the following:

1. 144, 729.
2. 225, 5^4.
3. $(2\,a)^3$, $24\,a^2c$.
4. $75\,x^2y$, $45\,xy^2$.
5. $361\,a^4b^2$, $38\,a^2b$, $114\,a^3b^3$.
6. $84\,z^3$, $54\,z^5x$, 9.
7. $3125\,a^5$, $625\,a^4$, $125\,a^3$.
8. $243\,a^5b^{10}$, $162\,a^4b$, $135\,b^3$.
9. $a^4b^3c^2z$, $a^3b^4cz^4$, ab^5cx^7.
10. $75\,c^2d$, $125\,cd^3$, $224\,ad$.

12. What does x^2y mean? How many factors are there : the expression? In x^2y, let $x = 3$, $y = 5$. What is the result

EXERCISE 6

In the following expressions substitute $x = 1$, $y = 0$, $z =$ $a = 3$, $b = 4$, and compute the value of the result:

1. $3\,a^2b^3 = 3 \cdot 3^2 \cdot 4^3 = 3 \cdot 9 \cdot 64 = 1728.$
2. $$\begin{aligned} 2\,xy + 4\,y^2 + 7\,x^2z &= 2 \cdot 1 \cdot 0 + 4 \cdot 0^2 + 7 \cdot 1^2 \cdot 5 \\ &= 0 \qquad + 0 \qquad + 35 \\ &= 35. \end{aligned}$$
3. ax^2z.
4. $35\,x + 3\,y - 6\,z$.
5. $10834\,xyz$.
6. $(5\,a + 2\,b)x$.
7. $2\,a(3\,b + 4\,z)$.
8. $7\,z(2\,x + 5\,y)$.
9. $\dfrac{y}{z}$.
10. $\dfrac{y + 5\,x}{z}$.
11. $(y + 5\,x)(y + 5\,x)$.
12. $(b - a)x$.

13. $(b-a+y)x$.

14. $\dfrac{b-a}{x}$.

15. $(3a+2b+2z)^2$.

16. $(15a-3b-6z)^3$.

13. The **parenthesis** as used in these examples denotes that the quantities inclosed are each subject to the same operation.

Thus, $(x+y)z$ means that the sum of x and y is to be multiplied by z, or that both x and y are to be multiplied by z and the sum of the products taken.

$(2+3)^2$ indicates that the sum of 2 and 3 is to be "squared," *i.e.*, used twice as a factor.

$(a+b)+(c+d)$ is read: the sum of c and d is to be added to the sum of a and b, or the sum of a and b plus the sum of c and d.

Likewise, $(a+b)-(c+d)$ indicates that the sum of c and d is to be subtracted from the sum of a and b.

The forms of *parenthesis* are ( ), the *brace* { }, the *bracket* [ ], and the *vinculum* $\overline{\quad}$. The vinculum is seldom used.

EXERCISE 7

1. What does $12x^2$ mean?
2. What does $(12x)^2$ mean?
3. What does $12(x)^2$ mean?

Perform the indicated operations:

4. $(18a+12a)+(5a+2a)$.
5. $(21a+2a)-(6a+3a)$.
6. $(21a-2a)+(6a-3a)$.
7. $(21a-2a)-(6a-3a)$.
8. $3(6x-2)+4(2x-1)$.
9. $5(a+3a)-2(a+2a)$.
10. $4[5(a+6a)]$.
11. $6[2(5x-3x)]-2[4(2x-x)]$.
12. $(5x+18y)-(2y+3y)$.
13. $(8x+12y+15z)-(x+2y)$.

Oral Review

The area of a rectangle equals the product of the base and altitude (length and breadth). Find the areas of the following rectangles:

	Length	Breadth		Length	Breadth
1.	*17″	6″	10.	19′	15′
$17 \cdot 6 = (10 + 7)6 = 60 + 42 = 102.$			11.	21′	24′
			12.	17′	18′
2.	18′	6′	13.	15′	13′
3.	18″	7″	14.	12″	24″
4.	19′	7′	15.	32″	21″
5.	20′	9′	16.	19′	28′
6.	22′	8′	17.	31′	22′
7.	15″	12″	18.	106″	12″
$15 \cdot 12 = 15(10 + 2) = 150 + 30 = 180.$			19.	115″	12″
			20.	106′	21′
8.	16′	12′	21.	112′	15′
9.	18″	12″	22.	1024′	4′

The area of a triangle is equal to one half the product of the base by the altitude. Find the areas of the following triangles:

	Base	Altitude		Base	Altitude		Base	Altitude
1.	16	10	6.	36	18	11.	37	19
2.	14	7	7.	44	22	12.	18	25
3.	24	8	8.	46	33	13.	25	16
4.	32	10	9.	48	24	14.	32	16
5.	24	12	10.	21	15	15.	17	21

* ″ indicates inches, ′ indicates feet.

The area of a circle is equal to 3.1416 (Abbreviation is π, pronounced pī) times the square of the radius.

Indicate the areas of the following circles:

(D represents diameter; R, radius; A, area.)

16. $D=12$, $A=\pi\cdot 6^2=36\,\pi$.

17. $R=18$, $A=\pi\cdot 18^2=324\,\pi$.

18. $R=20$, $A=$

19. $D=38$, $A=$

20. $D=46$, $A=$

21. $R=17$, $A=$

22. $R=27$, $A=$

23. $D=28$, $A=$

24. $D=56$, $A=$

25. $D=58$, $A=$

26. $R=26$, $A=$

27. $R=16$, $A=$

Find the side of a square whose area is:

28. 729 **29.** 841 **30.** 900 **31.** 441 **32.** 361 **33.** 529

Find the edge of a cube whose volume is:

34. 729 **35.** 512 **36.** 64 **37.** 1331 **38.** 343 **39.** 216

The area of a trapezoid is equal to the product of one half the sum of the lower base (B), and the upper base (b), by the altitude (a).

Find the areas of the following trapezoids:

	B	b	a
40.	10	5	6

$$\frac{(10+5)6}{2}=(10+5)3=45.$$

	B	b	a
41.	18	10	4
42.	22	16	8
43.	24	12	15
44.	13	13	16
45.	14	14	5
46.	9	18	8
47.	18	8	8
48.	16	14	9
49.	15	13	14
50.	24	15	6
51.	22	11	8
52.	15	17	17

Is there anything unusual about the trapezoids in examples *44 and 45?*

CHAPTER II

Equations

14. An **equation** is a statement that two quantities are equal. The sign of equality is $=$.

Thus, $x+3=5$ is read x plus 3 is equal to 5.

15. To *solve an equation* is to find the value or values of some letter involved in the equation which will satisfy the given equation.

16. When a number substituted for some letter in an equation makes the sides of the equation identical, the equation is said to be *satisfied.*

A number which satisfies an equation is called a **root** of the equation. The number is also said to be a *solution of the equation.*

Ex. $$x+3=5. \qquad (1)$$

Substitute $x=2$ in the equation.

Then, $$2+3=5.$$

The two sides or members of the equation are the same or identical.

The number on the left of the sign of equality is called the *first member* or side. The number on the right is the *second member* or side. Thus, in $x+3=5$, $x+3$ is the first member, and 5 is the second member.

17. The kinds of equations that concern us at present are:

The *equation of condition.*

The *identical equation* or *identity.*

The *geometric equation.*

18. An **equation of condition** is an equation that is satisfied *only by a definite set* of values.

E.g., in $x+3=5, x=2$ is the only value which can be found for x, which is a root of the equation. $x+3=5$ is therefore an equation of condition, the condition being that x must equal 2.

19. An **identity** is an equation which is always true for any specified values of the letters involved in it.

E.g., $2a=a+a$ is true for any finite value of a.

20. The **geometric equation** is an equation of two geometric figures. In general, the algebraic equation (§§ 18, 19) is assumed to be true, and if its roots satisfy it the statement of equality is verified. The geometric equation must usually be proved to be true (§ 70).

21. The operations used in equations are largely those of addition, subtraction, multiplication, and division.

22. The laws governing the use of these operations are a set of statements assumed to be true, and known as **axioms.**

23. **The Axioms**

1. *If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.*
2. *If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.*
3. *If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.*
4. *If equal numbers be divided by the same number, or equal numbers, the resulting numbers will be equal. It is not allowable to divide by* 0.
5. *Any number equals itself.*
6. *Any number equals the sum of all its parts.*
7. *Any number is greater than any of its parts.*
8. *Two numbers which are equal to the same number, or to equal numbers, are equal.*

The Use of Axioms

24. Ex. 1. Solve $2x+3=9$.

Subtract 3 from each member of the equation (Ax. 2).

$$2x = 9 - 3.$$

or $$2x = 6.$$

Divide each member by 2 (the coefficient of x) (Ax. 4).

Then, $$x = 3.$$

To verify this root, substitute 3 for x in the *given* equation.

$$2 \cdot 3 + 3 = 9.$$

The equation is satisfied, hence 3 is a root.

Ex. 2. Solve $y + \frac{11}{6} = \frac{2y}{3} + \frac{5}{2}.$

Multiply both members of the equation by 6, the L. C. M. of the denominators (Ax. 3).

$$6\left(y + \frac{11}{6} = \frac{2y}{3} + \frac{5}{2}\right),$$

or $$6y + 11 = 4y + 15.$$

Subtract $4y$ from each side (Ax. 2).

Then, $$2y + 11 = 15.$$

Subtract 11 from each side (Ax. 2).

Then, $$2y = 4.$$

Divide both sides by 2 (Ax. 4).

And, $$y = 2.$$

Verify by substituting $y = 2$ in the *given* equation.

$$2 + \frac{11}{6} = \frac{4}{3} + \frac{5}{2}.$$

Simplify each member.

$$\frac{12+11}{6} = \frac{8+15}{6}.$$

Hence, 2 is a root of the equation.

EXERCISE 8

Solve the following equations and verify each root:

1. $2y+7y+3=12.$ 2. $5x+7=3x+17.$

3. $\frac{5x}{6}-\frac{1x}{4}=3-2\frac{5}{12}$.

4. $6x+\frac{1}{4}=2x+\frac{5}{4}$.

5. $4x-2=6$ (Ax.).

6. $7x-4=x+14$.

7. $8z+2-3z-4=4z$.

8. $6u-4=3u+8$.

9. $y+\frac{3y}{5}+\frac{1}{4}=\frac{2y}{3}+\frac{61}{20}$.

10. The sum of two numbers is 9, and one is twice the other. Find the numbers.

Let $x =$ the smaller number.
and $2x =$ the greater number.
$2x + x =$ the sum of the numbers.
$9 =$ the sum of the numbers.
Then, $2x + x = 9$ (Ax. 8),
or $3x = 9$.
$x = 3$, the smaller number.
$2x = 6$, the greater number.

11. The difference between two numbers is 24, and the greater is four times the lesser. Find the numbers.

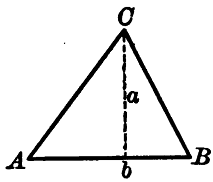
12. The sum of two numbers is 48, and the greater is four more than the lesser. Find the numbers.

13. A number is composed of two digits. The tens' digit is four times as great as units' digit, and the sum of the digits is 10. Find the number. (Exercise 2, Ex. 14.)

14. A number is composed of two digits. The tens' digit is three times the units' digit, and the number is 54 more than the sum of the digits. Find the number.

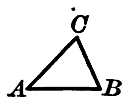
15. The distance around a rectangle is 120′. The length of the rectangle is 10′ more than the breadth. Find the dimensions.

16. Two rectangles each have an altitude of 8′, the sum of their areas is 256, and the difference of their lengths is 12′. Find length of each.



17. In a triangle ABC the area is 24, the altitude (a) is 8. Find the base (b).

18. In a trapezoid $ACDE$, the area is 120; the lower base B is four times the upper base b; the altitude is 8. Find the bases.



19. The sum of the angles of a triangle is equal to 180°. The angle at A is 20° more than the angle at B, and 50° more than the angle at C. Find the angles.

20. In the triangle ABC, $\angle A$ is twice $\angle B$, and $\angle C$ is equal to the sum of $\angle A$ and $\angle B$. Find the angles.

21. The sum of two numbers is 48 and one number is four times the other. Find the numbers. (Solve mentally.)

CHAPTER III

Positive and Negative Numbers. The Four Fundamental Operations

25. Positive and Negative Numbers. In addition to the numbers thus far used in computation, it is necessary in algebra to extend the idea of number somewhat farther. In many problems the numbers involved seem to have an opposite sense. For example: If a man has \$500 and owes \$300, the \$300 opposes the \$500. If a man is walking east, the opposite direction to that in which he is going is west. North and south are opposites. Temperatures above and below zero are opposites. To express this sense of opposition, positive (+) and negative (−) signs are used in mathematics. *E.g.*, if toward the right is +, then toward the left is −. If north is considered positive, south is negative. If assets are +, liabilities are −.

EXERCISE 9

1. If D is four units to the right of 0, and C two units to the left of 0, how far is it from C to D? Does that mean to you the difference between the positions of point C and point D? If A is −3 units from 0, and B is +5 units from 0, what is the distance from A to B?

2. A man has \$600 and owes \$300. How much is he worth?

3. A man has \$500 and owes \$700. How much is he worth?

4. A man has \$500 and owes \$500. How much is he worth?

5. Where on the Cleveland-Wooster railway line is a place -10 miles north of Cleveland?

6. A man goes 5 miles north of Cleveland, then 9 miles south. How many miles north of Cleveland is he? How many miles has he traveled? Draw a diagram showing his route and his last position.

7. If to the right is positive, measure $+6''$ from a point A. Call this point B. Measure $-9''$ from B. Call this point D. Where is D with respect to A?

8. Translate into English $(a^2 - b^2)^3$.
Write in symbols: The square of the result of the quotient of the sum of (a) and (b) by 2.

9. The temperature at 6.00 A.M. is $+14°$ and during the morning it grows colder at the rate of $4°$ an hour. Required the temperatures at 9 A.M., at 10 A.M., and at noon.

10. Find the numerical value of the following, when $a = 3$, $b = 5$, $c = 2$, $d = 4$: $\dfrac{a^3b}{2c^2} - \dfrac{ab^3}{4d^2}$.

11. Translate into English $\left(\dfrac{a+b}{2}\right)^2$.
Write in symbols: the cube of the result of subtracting the square of b from the square of a.

12. Find the numerical value of the following, when $x = \frac{1}{2}$, $y = \frac{2}{3}$, $z = \frac{1}{4}$: $4x^2 + (3y - 2z)^2$.

Addition

26. In § 8, we found the sums of similar terms, but in each instance the sum was positive. A negative sum may arise from the addition of a positive and a negative number. For instance, in example 6, exercise 9, we are adding -9 miles to $+5$ miles. The result is -4 miles. That is, *the addition of a negative number to a positive number tends* to lessen the numerical value

of the sum, annul it, or change it from positive to negative. If a man has \$400 and owes \$400, the sum of his assets and liabilities is 0. If he has \$400 and owes \$700, the sum of his assets and liabilities is $-$ \$300. That is, he owes \$300 more than he has assets. The sum of two negative numbers is negative. It is seen in these results that when the sum of a positive and a negative number is found, the result takes the sign of the greater *absolute value.*

27. The **absolute value** of a number is its value regardless of sign. For example, the absolute value of -6 is 6.

28. If no sign is placed before a number it is regarded as **positive.** A negative sign *must never be omitted.*

EXERCISE 10

Find the sums of the following:

1. $\begin{array}{r} +5 \\ \underline{+3} \end{array}$ 2. $\begin{array}{r} +5 \\ \underline{-3} \end{array}$ 3. $\begin{array}{r} -5 \\ \underline{+3} \end{array}$ 4. $\begin{array}{r} -5 \\ \underline{-3} \end{array}$

5. $3a^2$ and $-4a^2$.

6. $7ab$, $-5ab$.

7. $-8x$, $-5x$.

8. $9a^2$, $-6a^2$, $-3a^2$, $-12a^2$, $-5a^2$.

9. $3a$, $-5a$, $+6a$, $-4a$.

*10. $7a^2$, $4x$, $+3a^2$, $+2x$, $+x$.

11. $12x^2y$, $6xy^2$, $4x^2y$, $-2x^2y$, $+11xy^2$.

12. $15x^3$, $-15x^2$, $+15x$, $+5x^2$, $-5x$, $-9x^3$, $-9x^2$, $-9x$.

13. $4x+3y-z$, $-2x+y+4z$, $-x-3y-2z$.

14. $5(a+b)$, $-2(a+b)$, $6(a+b)$, $-9(a+b)$.

* Write similar terms (§ 6) in the same column. Make as many columns as you have different kinds of terms, forming the whole into one problem, by *using* + *and* − *signs.* (See example 2, § 8.)

15. $4(x+y)-3(x-2y),\ -3(x+y)+2(x-2y),\ 2(x+y)+(x-2y)$.

16. $2(a+b)^2-6(a+b)+1,\ -5(a+b)^2+3(a+b)-6,\ (a+b)^2-(a+b)+2$.

17. $\frac{2}{3}x-\frac{3}{5}y+\frac{1}{2}z,\ \frac{3}{4}x+\frac{2}{3}y-\frac{1}{4}z,\ -\frac{1}{2}x+\frac{3}{10}y-\frac{1}{2}z$.

18. $\frac{7}{8}a-\frac{5}{4}b+\frac{2}{3}c,\ \frac{1}{16}a-\frac{1}{8}b-\frac{2}{5}c,\ -\frac{5}{8}a+\frac{7}{8}b-\frac{7}{10}c$.

Find the value of the following sums when $x=\frac{1}{2}$, $y=+\frac{1}{4}$, $z=\frac{1}{3}$, $a=-2$, $c=\frac{1}{5}$, $b=2$:

19. $\frac{1}{2}a+\frac{1}{2}b-c,\ +a-\frac{1}{4}b-\frac{2}{3}c,\ 5a-\frac{2}{5}b+2c$.

20. $2(a+b)^2-3(b-c)^2+(a-b),\ -500(a+b)^2+5(b-c)^2+(a-b)$.

21. $5xy-5x^2y-5xy^2,\ \frac{1}{2}xy+\frac{8}{3}x^2y$.

22. $7(x+2y)-7(x-2y),\ -31(x+2y)+\frac{3}{7}(x-2y)$.

23. $12yz-8xy+\frac{1}{4}a+\frac{5}{2}bc$.

24. $\frac{2}{3}a-\frac{3}{4}b+\frac{5}{2}c,\ +\frac{3}{2}a-\frac{1}{2}b+25c$.

In each of the following examples, add corresponding members of the two equations to find x:

25. $x+y=8$,
$\underline{x-y=4}$. When x is found, can you find y?

26. $2x+3y=8$,
$\underline{x-3y=-5}$. Find y also.

27. $4x-2y=2$,
$\underline{3x+2y=12}$. Find y also.

28. Verify your results in examples 25–27 by substituting the values found for x and y in the given equations.

Subtraction

29. (*a*) What number added to 9 gives 7?
(*b*) What number added to 9 gives 11?

(*c*) What number added to 9 gives 0?
(*d*) What number added to 9 gives -12?

In each of the above examples we have the sum of two numbers and one of the numbers given to find the other number.

The **Minuend** is the sum of two numbers.

The **Subtrahend** is a given number.

The **Difference** is a required number when the minuend and subtrahend are known.

Subtraction is the process of finding what number added to the subtrahend produces the minuend.

In example (*a*), 7 is the minuend, 9 is the subtrahend, -2 the difference. In subtraction, the sum of the subtrahend and the difference must equal the minuend. This fact enables us to check our result.

EXERCISE 11

1. Subtract -3 from 5.

Here 5 is the sum of the numbers. -3 is one of the numbers. Our problem is: What number added to -3 gives 5. This number is evidently 8. That is, the *difference* is 8.

Check: $8+(-3)=$ sum of the numbers.
$5=$ sum of the numbers.

The written work stands:

$$\begin{array}{r} 5 \\ \underline{-3} \\ 8 \end{array}$$

2. From $-8a$ take $5a$:

$$\begin{array}{r} -\ 8a \\ \underline{+\ 5a} \\ -13a \end{array}$$

Perform the following subtractions:

3. $\begin{array}{r} 7x^2y \\ \underline{-3x^2y} \end{array}$ **4.** $\begin{array}{r} 25x \\ \underline{13x} \end{array}$ **5.** $\begin{array}{r} 13x \\ \underline{25x} \end{array}$ **6.** $\begin{array}{r} -25x \\ \underline{-13x} \end{array}$ **7.** $\begin{array}{r} -25x \\ \underline{-13x} \end{array}$

8. From $-13\,ab$ take $24\,ab$.

9. From $a+b+c$ take $a-b-2\,c$.

(Note that there are three subtraction examples in this example, one for each column.)

10. Subtract $8\,x-2\,y+z$ from $10\,x-y-3\,z$.

11. Subtract $2\,a^2-3\,a+1$ from $5\,a^2-3\,a-1$.

12. Subtract $10\,a^2+5\,ab-9\,b^2$ from $2\,a^2-10\,ab+8\,b^2$.

13. From $x^3+3\,x^2+3\,x+1$ take $x^3+2\,x^2+2\,x+1$.

14. From $x^3+3\,x^2+3\,x+1$ take $x^3-3\,x^2+3\,x-1$.

15. From $a^2+2\,ab+b^2$ take $a^2-2\,ab+b^2$.

16. From $5\,x^2$ take $5\,y^2$.

17. From $6\,x^2-5\,x+4$ take $12\,x^2-9$.

18. From $3\,x^2$ take $-4\,x^3+3\,x^2-2\,x+1$.

19. Subtract a from 0.

20. Subtract $9\,x^2+9\,y+9$ from 0.

Note that in each of the above examples the difference is the same as if we had changed the sign of the subtrahend and added the result to the minuend.

We may then use the following rule for subtraction. *Change the sign of each term of the subtrahend and proceed as in addition. (The change of sign must be made mentally.)*

EXERCISE 12

1. What must be added to $9\,x^2+9\,y+9$ to give 0?

2. From $x^2+4\,x+4$ take $x^2-4\,x+4$.

3. From the sum of $a^2+2\,ab+b^2$ and $a^2-2\,ab+b^2$ take the difference between $a^2+2\,ab+b^2$ and $a^2-2\,ab+b^2$.*

* *In this text* the difference between a and b means the remainder obtained *by subtracting* b *from* a.

4. Subtract $2a^3 + a^2b + ab^2 + 2b^3$ from the sum of $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 - 3a^2b + 3ab^2 - b^3$.

5. Subtract $x^2 + 3x - 4$ from x^3.

6. Subtract $b + c$ from a.

7. Subtract $4x^2 - 7x - 9$ from 0.

8. What shall we add to $7x^2 - 12x + 5$ to produce 0?

9. $[5x^3 - (3x + 2)] - [6x^3 + (4x - 11)]$.

In examples of this type perform the indicated operations *one at a time, beginning with the inner parenthesis.* First subtract $3x + 2$ from $5x^3$, then add $4x - 11$ to $6x^3$. This gives

$$[5x^3 - 3x - 2] - [6x^3 + 4x - 11].$$

Subtracting the second trinomial from the first, we have

$$-x^3 - 7x + 9.$$

Translate each example into English before solving:

10. $\{5a - [2a + (4a + 1)]\}$.

11. $[12x^2 - \{-7x^2 - (5x^2 + 6x - 3)\}]$.

12. $[6(5a + 3) + 5(2a + 7)] - (6a + 53)$.

13. $[25m - (2m + 3)] - [-10m - (6m - 7)]$.

14. $[x^3 - (3x^2y - 3xy^2 - y^3)] - [x^3 + (-3x^2y + 3xy^2 - y^3)]$.

15. $(12a + 3b + 2c) - (5a - 2b + 6c) - (10a - b - 6c)$.

16. $4(x + y) - [6(x - y)] + (x - 2y)$.

17. There are two numbers, x and y, whose sum is 17 and whose difference is 1, x being the greater number. Find the numbers.

By the conditions, $\quad x + y = 17 \quad (1)$

$\quad x - y = 1 \quad (2)$

Adding (1) and (2), $\quad 2x = 18$

$\quad x = 9.$

Subtracting (2) from (1), $\quad 2y = 16.$

$\quad y = 8.$

18. Given two numbers, x and y, such that the second number added to twice the first is equal to 12, and the difference between twice the first number and the second number is 8. Find the numbers.

19. $5x+2y=19,$ (1)
$\underline{5x-2y=11.}$ (2) Find x and y.

20. $x+3y=16,$ (1)
$\underline{x-3y=-14.}$ (2)

21. $2x+3y=17,$ (1)
$\underline{x-3y=-14.}$ (2) After x is found, obtain the value for y by substituting the value of x in equation (1).

22. Find the value of the difference between $5x^3+3x^2y-5a^2$ and $2x^3-3x^2y-2a^2$, when $x=5$, $y=\frac{1}{6}$, $a=3$.

Multiplication

30. In addition we learned that

$$5+5+5+5=5\cdot 4=20$$

Also that

$$(-5)+(-5)+(-5)+(-5)=(-5)(4)=-20.$$

This last shows that the product of a negative number by a positive number is negative.

In arithmetic, it does not matter in what order the factors are used, *e.g.*, $5\cdot 3=3\cdot 5$.

We shall assume that this law holds true in algebra.

$$\text{Then, } (-5)(4)=(4)(-5)=-20.$$

That is, the product of a positive number by a negative number is *negative*. It seems then that multiplication by a negative number gives to the product a sign opposite to that of the multiplicand.

Ex. $(-8)\cdot(-5)=+40.$

31. These rules for signs follow:

The product of two numbers of like sign is positive.

The product of two numbers of unlike sign is negative.

32. Since division is the inverse of multiplication, the same sign rules hold; namely,

Like signs give plus and unlike signs give minus.

Namely, when the terms have like signs, the quotient is positive; when unlike, the quotient is negative.

Division may be regarded as an application of the method of subtraction.

Ex. Divide 28 by 7.

This is equivalent to successively subtracting 7 until no remainder or a remainder less than 7 remains.

$$28 - 7 = 21, \quad 21 - 7 = 14, \quad 14 - 7 = 7, \quad 7 - 7 = 0.$$

Which shows that 7 is contained in 28 four times without a remainder, or that 7 is an integral factor of 28.

Regarding division as the inverse of multiplication is the more general method. With this understanding we have the following definitions.

33. **Division** is the process of finding one of two factors when their product and one of the factors are given.

The **Dividend** is the product of the factors.

The **Divisor** is the given factor.

The **Quotient** is the required factor.

It is evident from these definitions that the product of the divisor by the quotient is equal to the dividend. By the use of this principle the accuracy of the result of the division may be verified.

Give *sign rules* for addition, subtraction, multiplication, and division.

EXERCISE 13

Find the product of the following (first determine the sign of the product):

1. $(4)(-3)$.
2. $(-9)(4)$.
3. $(-9)(-8)$.
4. $(+10)(+10)$.
5. $(-10)(-10)$.
6. $(-7)(+9)$.

Find the following quotients:

7. $(+12) \div (+4)$.
8. $(-12) \div (-3)$.
9. $(-15) \div (+5)$.
10. $(-20) \div (-4)$.
11. $(-112) \div (16)$.
12. $(144) \div (-18)$.

13. The area of a rectangle is 212, the base is 17. Find the altitude.

14. The area of a triangle is 180, the base is 20. Find the altitude. Can you construct the triangle? Why?

15. The area of a rectangle is 24, the altitude is -4. What is the base? How do you account for these negative results? Draw this rectangle. (§ 25.)

34. In multiplication it is convenient to show the number of times a quantity is used as a factor by means of a symbol placed at the right and above a quantity.

This symbol showing to what power the number is to be raised is the *exponent* of the number.

Ex. 1. $x \cdot x \cdot x = x^3$. $(2x)(2x)(2x)(2x) = (2x)^4 = 16x^4$.

Similarly, $x \cdot x \cdot x \cdots n$ factors $= x^n$.

Ex. 2. $x^2 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$.

$$\text{And } x^n \cdot x^m = (x \cdot x \cdot x \cdots n \text{ factors})(x \cdot x \cdot x \cdots m \text{ factors})$$
$$= x \cdot x \cdot x \cdots m + n \text{ factors} = x^{m+n}.*$$

Then in multiplying two *like* letters give to their product an exponent equal to the sum of their exponents in multiplicand and multiplier.

* ... is read, "and so on to."

We see that multiplication is simply combining the factors of the multiplicand and multiplier into one product; *e.g.*, $12 \cdot 18$ means $(2^2 \cdot 3)\ (3^2 \cdot 2)$ or $2^3 \cdot 3^3$, the product containing all the factors of both multiplicand and multiplier. Similarly, $24\,a^2b \cdot 15\,a^3$

$$= 2^3 \cdot 3 \cdot a^2 \cdot b \cdot 3 \cdot 5 \cdot a^3 = 2^3 \cdot 3^2 \cdot 5 \cdot a^5 \cdot b = 360\,a^5b.$$

35. Since division is the inverse of multiplication, the exponent rule for division is the reverse of that in multiplication.

In dividing one like letter by another, give to the quotient an exponent equal to the exponent of the dividend minus the exponent of the divisor.

Ex. 1. Divide x^3 by x^2.

$$x^3 = x \cdot x \cdot x,$$
$$x^2 = x \cdot x.$$

Then, $(x \cdot x \cdot x) \div (x \cdot x) = x$,

or $x^3 \div x^2 = x^{3-2} = x^1$.

When the exponent is 1, it is not expressed.

Thus, $x^3 \div x^2 = x$.

Ex. 2. Divide a^{15} by a^{12}.

$$a^{15} \div a^{12} = a^{15-12} = a^3.$$

EXERCISE 14

Find the following indicated products:

(Translate each example into English before solving.)

1. $x^5 \cdot x^3$.
2. $15\,x^2 \cdot 5\,x^3$.
3. $8\,m \cdot 18\,m^4$.
4. $(15\,a)^3(2\,a)$.
5. $(x+y)^2(x+y)^3$.
6. $12(a+b) \cdot 3(a+b)$.
7. $-19(a-b)^4 \cdot 15(a-b)^5$.
8. $(12\,c^3) \cdot (-4\,c^5)$.
9. $a^3b \cdot a^4b^2$.
10. $-15\,a^2b^2 \cdot 21\,a^3b^5$.
11. $(-13\,a^2b^6c)(-13)$.
12. $(18\,c^3d)(-19\,c^4d^2)$.
13. $(27\,ab^2xy)(-14\,a^5bc)$.
14. $(-21\,x^2yz)(-21\,x^3y^3z^2)$.

15. $12(a+b)$ by $12(a+b)^2$ by $12(a+b)^3$. What power of 2 is in your product? What power of 3? What power of $(a+b)$? What then are the prime factors of your product?

Indicate the prime factors of these products:

16. $(18a^3b^4)(18ab^2)$.

17. $(-24p^2q^4r)(-24p^2q^4r)$.

18. $32(a+x)^5$ by $64(a+x)^6$.

19. $27(c+d)^3$ by $9(c+d)^2$.

20. $125(x-y)^3$ by $25(x-y)^2$.

Find these quotients and verify your results:

21. $x^8 \div x^5$.

22. $x^4 \div x$.

23. $72x^6 \div 9x^2$.

24. $441a^6 \div 21a^3$.

25. $(-24x^5) \div (-3x^2)$.

26. $(576a^7b^4) \div (-24a^5b^2)$.

27. $729c^4x^7 \div 9cx^3$.

28. $(c-d)^{10} \div (c-d)^4$.

29. $27(a+b)^3 \div 9(a+b)^2$.

30. $a^5b^3c^2 \div a^2bc$.

31. $2^5 \cdot 5^3 \cdot 3^2 \div 2^2 \cdot 5 \cdot 3$.

Are examples 30 and 31 similar?

32. Divide $5^4 \cdot 3^5 \cdot 2^4 a^7b^6y^4$ by $5^3 \cdot 2^4 a^7b^6y^2$.

Factor the dividend and divisor. Then divide as in example 32:

33. $162x^4y \div 54xy$.

34. $-135c^7x^5 \div 15cx^5$.

35. $-210x^4y^3z^2 \div -14x^4y^3$.

36. The area of a rectangle is $128(x+y)(c+d)$, its altitude is $8(c+d)$. What is its base? What are its dimensions if $x=1$, $y=2$, $d=4$, $c=-3$?

37. The area of a rectangle is $384x^5yc^2$, its altitude is $16x^2c$. Find its base. What are its dimensions if $x=-2$, $y=1$, $c=-2$? Also if $x=2$, $y=1$, $c=2$? Draw these rectangles, measuring from the same starting point for each.

35. The base of a triangle is $27\,a^3c^4$, the area is $243\,a^4c^5$. Find the altitude. What are the dimensions when $a = -3$, $c = \frac{1}{3}$?

36. Give the sign rules for addition, subtraction, multiplication, division.

Give the exponent rules for multiplication and division.

Supplemental Applied Mathematics

A **decimal fraction** is a fraction whose denominator is a power of 10. This denominator may be written, or simply expressed by the relative position of the decimal point. The factors of 10 are 2 and 5. The factors of 100 are two 2's and two 5's, that is, $100 = 2^2 \cdot 5^2$. Likewise $1000 = 2^3 \cdot 5^3$. Every power of 10 is made up of an equal number of 2's and 5's used as factors. Therefore, to reduce a common fraction to a *pure decimal* one must multiply both numerator and denominator by such number as will produce an equal number of 2 and 5 factors only in the denominator.

Ex. **1.** Reduce $\frac{3}{4}$ to a decimal.

$\frac{3}{4} = \frac{3}{2^2}$. Two 5's are lacking. Multiplying both numerator and denominator by 5^2 or 25, we have $\frac{75}{100}$, or, expressed decimally, .75.

Ex. **2.** Reduce to .1875 a common fraction:

$$.1875 = \frac{1875}{2^4 \cdot 5^4} = \frac{3 \cdot 5^4}{2^4 \cdot 5^4} = \frac{3}{16}.$$

3. Reduce to decimals: $\frac{1}{2}$, $\frac{3}{5}$, $\frac{7}{20}$, $\frac{17}{40}$, $\frac{7}{8}$, $\frac{19}{50}$.

4. A machinist has a set of drills marked .1250; .9375; .0625; .03125; .3125; .8750. He does not recognize them as readily as if they were in 8ths, 16ths, 32ds, 64ths. Reduce them to such notation.

5. Reduce the following sizes of drills to their decimal equivalents: 1–8, 3–32, 3–16, 5–16, 7–16, 9–16, 11–16, 13–16, 19–16.

In all computation in science and shop practice the decimal plays an important part.

Whether multiplication and division is carried on directly or by means of tables, the computer must *know* at a glance where the decimal point is to be placed.

Multiplication of Decimals

Decimals are multiplied as simple numbers if one remembers the decimal composition of our system and the part the position of the digit plays in the formation of a number. In 145, § 1 and exercise 1, the 1 has 100 times the value it would have if written in units' column where the 5 now is. The 4 has ten times the value it would have if in units' column. Then, multiplication by a digit in hundreds' column has 100 times the effect of multiplication by the same number in units' column. A similar statement holds for digits in tenths' and hundredths' columns, each move to the right decreasing the value of a digit ten times.

In multiplication we begin at the left.

Ex. 6. Multiply 24 by 36.

$$\begin{array}{r} 24 \\ 36 \\ \hline 720 \\ 144 \\ \hline 864 \end{array}$$

We multiply first by 3. Since this figure is in tens' column, it has ten times the value of a figure in units' column, and our product moves one place to the left. That is, we are really multiplying by 30 units. The *units' column*, not the decimal point, is the *dividing line.*

Ex. 7. Multiply 23.2 by 2.4.

$$\begin{array}{l} 23.2 \\ \ \ 2.4 \\ \hline 46.4 \\ \ \ 9.28 \\ \hline 55.68 \end{array}$$

Always keep the decimal points in the same vertical column. When multiplying by 4, the multiplier was *one place* to the *right* of units, and the product $\frac{1}{10}$ as large as the product by the same figure if in units' place. This shifted our product, 9.28, one place to the right.

Always begin multiplication at the left.

Find the following products:

8. $17 \cdot 26$.

9. $324 \cdot 14$.

10. $324 \cdot 324$.

11. $216 \cdot 36$.

12. $9^2 \cdot (3.1416)$.

13. $16^2 \cdot (3.1416)$.

14. $24 \cdot 62.5 \cdot (.92)$.

15. $(.8)^3$.

16. $(1.2)^3$.

17. $(2.7)^2$.

18. $(.27)^2$.

19. $(.09)^3$.

20. $24.21\ (.32)$.

21. $1.875 \cdot 16.2$.

22. $1.875 \cdot 1.62$.

23. $1.875\ (.162)$.

24. $18.75\ (1.62)$.

25. $1.112\ (.99)$.

26. $.6425\ (.0125)$.

How much multiplying is necessary in example 27, if the result is correct to three decimal places?

27. $4.261\ (.7854)$.

28. $32.15\ (.625)$.

Division of Decimals

Here we must reverse our work of multiplication. The first figure of the divisor and its distance from units' column determine the position of the decimal point with respect to the first figure of the quotient. If the first figure of the divisor is in tens' column, the quotient will be ten times as small (or one tenth as much) as if the divisor were units. The first figure of the quotient therefore moves one place to the right of the first figure of the dividend.

Ex. **29.** Divide 144. by 72.

$$\begin{array}{r|l} & \underline{02.} \\ 72. & 144. \\ & \underline{144.} \end{array}$$

Note that 7 is not contained in 1, the first figure of the quotient; therefore a *zero is written* for the first figure of the quotient. Since this zero

is to the left of an integer, it has no value and may be disregarded. Until the beginner has the placing of the first figure of the quotient well in mind, this zero should be employed.

Ex. **30.** Divide 324. by 18.

```
      18.
18. | 324.
    | 18 .
      144.
      144.
```

Ex. **31.** Divide 2.446 by 3.2.

```
      0.76 +
3.2 | 2.446
    | 2.24
       .206
       .192
       .014
```

Ex. **32.** Divide 2.446 by .32.

```
      07.6+
.32 | 2.446
    | 2.24
       .206
       .192
       .014
```

Ex. **33.** Divide 2.446 by 32.

```
       .076+
32. | 2.446
    | 2.24
       .206
       .192
       .014
```

Note that the position of the *first figure of the divisor alone* controls the position of the first figure of the quotient.

Find the following quotients. (Use three decimal places if *the quotient is not* exact.)

34. $31.5 \div 24.6$.

35. $6.4 \div 1.6$.

36. $6.4 \div .16$.

37. $6.4 \div .016$.

38. $6.4 \div 16$.

39. $.225 \div 15$.

40. $.225 \div 1.5$.

41. $.225 \div .15$.

42. $109.624 \div 3.86$.

43. $125.664 \div 3.1416$.

44. $251.328 \div 3.1416$.

45. $442 \div .09$.

46. $2.24 \div 22.4$.

47. $361. \div .19$.

48. $5.29 \div .23$.

49. $7.29 \div 2.7$.

50. $7.29 \div .27$.

51. $.0289 \div 1.7$.

52. $.0289 \div .17$.

53. $.0289 \div 17$.

Locate by inspection the first figure in each of the following quotients. (Remember that the number of places in the quotient does not in any way depend upon the number of *figures* in the divisor.) The *position* of the first figure in the divisor is all that one needs consider.

For example, in dividing 8.432694 by .3419768321, the first figure 2 takes the same position as if we were dividing by .3, namely, in tens' column.

$$\begin{array}{r|l} & 2\ . \\ \hline .3419768321 & 8.432694 \end{array}$$

54. $48.36579 \div 4.6293251$.

55. $4.836579 \div 4.6293251764$.

56. $4.836 \div 4.6293251764847$.

57. $.2793 \div .217398$.

58. $.2793 \div .0217398$.

59. $31.84 \div 309.7$.

60. $34.76 \div 38$.

61. $.026947321 \div .41976384$.

62. $.178643791 \div 2.9$.

63. $.243 \div .0986432791$.

64. $2.43 \div .986432791$.

65. $.0243 \div 9.864327915$.

66. Multiply 16346.2′ by .00019, and show that the product is in miles.

67. Divide 3 cubic feet by .00058, and show that the quotient is cubic inches.

68. Rolled oats requires $1\frac{3}{4}$ hours cooking on a range. If a fireless cooker is used, 15 minutes cooking on the range is sufficient. How much fuel is saved by using the fireless cooker, if 8.6 cu. ft. of gas per hour are consumed by a gas burner, gas costing 76¢ per thousand cubic feet?

69. Wheatena, or cream of wheat, requires $\frac{3}{4}$ hours cooking on a range, or 15 minutes cooking on a range for a fireless cooker. How much fuel is saved by using the fireless cooker, gas burner and price same as in Ex. 68?

70. Corn meal mush should be cooked for three hours on a range, or for 15 minutes if a fireless cooker is used. How much fuel is saved by using a fireless cooker?

71. Uncooked rice contains 79 % carbohydrates, while boiled rice contains 24.4 %. How much carbohydrate is lost from one pound of rice by boiling?

72. Rice boils in 20 minutes, using a gas burner consuming 8.6 cubic feet per hour. Steamed rice is cooked on the same burner for 5 minutes, and on the simmering burner for 45 minutes, the latter consuming 3.6 cubic feet per hour. What is the difference in the cost of cooking?

73. Rice swells $3\frac{7}{8}$ times by boiling. If a recipe for pudding calls for one quart boiled rice, how much uncooked rice should be used?

CHAPTER IV

Inequalities. Simultaneous Equations

37. The sign of inequality is $>$ or $<$, the opening being toward the greater number.

Thus $9 > 5$ is read, "9 is greater than 5."

$4 < 7$ is read, "4 is less than 7."

$-5 > -9$ is read, "-5 is greater than -9."

38. Two inequalities are in the same sense when the first member of each is greater or less, respectively, than the second member.

39. If equal numbers be added to or subtracted from both sides of an inequality, the resulting inequality is said to continue in the same sense. That is, the inequality sign is not reversed.

Ex. $9 > 5$. Subtract 3 from each side. Then

$$9 - 3 > 5 - 3$$

or $$6 > 2.$$

40. If all of the signs of an inequality are changed, the inequality does not continue in the same sense.

That is, if all the signs of inequality are changed, the sign of inequality must be reversed.

41. An inequality continues in the same sense after being multiplied or divided by a positive number.

Ex. 1. $2x > 10$.

Dividing both sides by 2, we have

$$x > 5.$$

Ex. 2. $$2x + y > 8, \quad (1)$$
$$y = 2. \quad (2)$$

Subtract (2) from (1), $2x > 6$ (3) (§ 39)
Divide (3) by 2, $x > 3$ (4) (§ 41)
That is, to satisfy the above conditions, x must be greater than 3.

Transposition

42. In § 24 we solved an equation,

$$2x + 3 = 9$$

by subtracting 3 from each side, by Axiom 2, then dividing both sides of the resulting equation by the coefficient of the unknown number.

By a method called transposition, it is possible to simplify the first operation.

Ex. 1. Solve for x, $mx + c = b$. (1)

Subtracting c from each side, $mx = b - c$. (2)

Note that the equation (2) is the same as if we had removed $+ c$ from the first member of the equation (1) and placed it in the second member with its sign changed.

Such change is called transposition.

This rule follows: *A term may be transposed from one member of an equation to the other if its sign is changed.*

Then, in solving, $mx + c = b$.
Transposing c, $mx = b - c$,
and dividing both members of the equation by m, the coefficient of x,

$$x = \frac{b - c}{m}.$$

Ex. 2. $ax + 7 = 13 - 2ax$. (1)

Transposing the unknown term $-2ax$ to the first side, and $+7$ to the second side, we have $ax + 2ax = 13 - 7$, (2)

$$3ax = 6, \quad (3)$$
$$x = \frac{6}{3a} = \frac{2}{a}. \quad (4)$$

To verify this value of x, substitute the value of x in (1).

$$a\left(\frac{2}{a}\right)+7=13-2\,a\left(\frac{2}{a}\right),$$

or

$$2+7=13-4.$$

$\therefore$ $9=9.$ $\therefore$ is the abbreviation for *hence.*

Simultaneous Equations

43. In examples of exercise 8 and the last ones of exercise 10, we note that in the former exercise one unknown number is involved in the equation, while in the latter the values of two unknowns must be found.

The method for the solution of equations in two unknowns is to find some way of combining the two equations into one equation containing one unknown, and solving this resulting equation.

Such process is called **elimination.**

Two equations in two unknowns are necessary because one equation in two unknowns has no definite solution, but has an indefinite number of solutions.

E.g., in
$x+y=6$, if $x=0$, $y=6$; $x=1$, $y=5$; $x=2$, $y=4$; $x=3$, $y=3$; etc., any number of pairs of values satisfying the given equation.

44. Note that in this equation y changes value when x changes. This is because x and y are so related that their sum must always be 6. Such values must be given x and y that the equation must be kept in balance; that is, must be kept an equation.

The unknown numbers which are subject to change of value in an equation are often called **variables.**

45. The usual methods of elimination are addition or subtraction, substitution, and comparison.

46. **First Method. Addition or Subtraction.** In this method one or both *equations* are multiplied in such a manner that the

coefficients of the same unknown in both equations become identical. One unknown number may then be eliminated by addition or subtraction of the corresponding sides of the equation.

Ex. 1.
$$2x + y = 9, \quad (1)$$
$$3x - 2y = 10. \quad (2)$$

Multiply (1) by 2. $4x + 2y = 18$ (3)

Add (2) and (3), $3x - 2y = 10$ (2)

Whence, $7x = 28$ (4)

and $x = 4.$

Substitute $x = 4$ in (1), $8 + y = 9.$ (5)

Transposing (§ 42) $+8$ in (5), we have,

$$y = 9 - 8,$$

or, $y = 1.$

Check these values in (2),

$$3 \cdot 4 - 2 \cdot 1 = 10,$$
$$12 - 2 = 10.$$

Ex. 2.
$$8x - 5y = 31, \quad (1)$$
$$12x + 13y = -15. \quad (2)$$

Multiply (1) by 3, and (2) by 2,

$$24x - 15y = 93 \quad (3)$$
$$24x + 26y = -30 \quad (4)$$

Subtract (4) from (3), $-41y = 123.$ (5)

Divide (5) by -41, the coefficient of y.

Whence, $y = -3.$ (6)

Substitute value of y in (1),

$$8x - 5(-3) = 31,$$
$$8x + 15 = 31. \quad (7)$$

Transposing (§ 42) $8x = 31 - 15.$

$$8x = 16. \quad (8)$$

Whence, $x = 2.$

Check in (2), $12 \cdot 2 + 13(-3) = -15.$

$$24 - 39 = -15.$$

EXERCISE 15

Solve the following equations and verify results:

1. $3x - 4y = 1,$
   $4x + 3y = 18.$

2. $5t + 7u = 29,$
   $5t - 7u = 1.$

3. $3v - 5z = 12,$
$\underline{8v + 10z = -38.}$

4. $7y + 8x = -31,$
$\underline{12x - 3y = -3.}$

5. $4x + 5y = 29,$
$\underline{6x + 7y = 41.}$

6. $\frac{1}{3}x - \frac{1}{2}y = 1,$
$\underline{\frac{1}{3}x + \frac{1}{2}y = 5.}$

7. $\frac{1}{4}x + \frac{3}{2}y = \frac{15}{2},$
$\underline{\frac{3}{8}x - \frac{3}{4}y = -\frac{3}{4}.}$

8. $\frac{1}{2}x + \frac{1}{5}y = 6,$
$\underline{\frac{1}{8}x + \frac{4}{5}y = \frac{21}{4}.}$

9. $28x - 45y = -17,$
$\underline{35x - 27y = 8.}$

10. $36x + 25y = 11,$
$\underline{24x + 55y = -31.}$

47. Second Method. Substitution.

Ex. Solve $\quad 2x + y = 9, \quad (1)$
$\underline{3x - 2y = 10.} \quad (2)$

Solve (1) for y, $\quad y = 9 - 2x. \quad (3)$
Show that this is transposition.
Substitute the value of y in (3) in (2).
$3x - 2(9 - 2x) = 10. \quad (4)$
Then, $\quad 3x - 18 + 4x = 10$
(How do you account for the signs of 18 and $4x$?)
Transposing and collecting, $\quad 7x = 28.$
$x = 4. \quad (5)$
Substituting (5) in (3), $\quad y = 9 - 2 \cdot 4,$
or, $\quad y = 1.$
Check: $\quad 2 \cdot 4 + 1 = 9.$

EXERCISE 16

Solve by method of substitution (verify results):

1. $4x + y = 6,$
$\underline{3x + 4y = 11.}$

2. $12m - 7v = -2,$
$\underline{11m - 12v = -13.}$

3. $4R - 6S = -1,$
$\underline{8R + 3S = 3.}$

4. $4R - 6S = 3,$
$\underline{8R + 3S = 1.}$

5. $15y - 8z = 11,$
$\underline{35y + 6z = -32.}$

6. $10t - 8u = 7,$
$\underline{6t + 16u = -1.}$

48. Third Method. Comparison.

Solve each equation for the *same variable* and equate the values thus found.

Ex. 1.
$$2x + y = 9, \qquad (1)$$
$$3x - 2y = 10. \qquad (2)$$

Solve (1) for y, $y = 9 - 2x$. (3)

Solve (2) for y, $y = \frac{3x - 10}{2}$. (4)

Equate the values of y found in (3) and (4) (Ax. 8).

$$9 - 2x = \frac{3x - 10}{2}. \qquad (5)$$

Multiply (5) by 2, $2\left(9 - 2x = \frac{3x - 10}{2}\right)$,

or, $18 - 4x = 3x - 10$. (6)

Solving (6), $x = 4$.

Substitute in (3), $y = 9 - 8$
$= 1$.

Check as before.

Ex. 2.
$$8x - 5y = 31, \qquad (1)$$
$$12x + 13y = -15. \qquad (2)$$

Solve (1) for x, $8x = 5y + 31$,

$$x = \frac{5y + 31}{8}. \qquad (3)$$

Solve (2) for x, $12x = -13y - 15$,

$$x = \frac{-13y - 15}{12}. \qquad (4)$$

Equate the values of x in (3) and (4) (Ax. 8).

$$\frac{5y + 31}{8} = \frac{-13y - 15}{12}. \qquad (5)$$

Multiply (5) by 24, a multiple of the denominators (Ax. 3).

Then, $15y + 93 = -26y - 30$. (6)

(See § 46, and note that the terms of (6) are the same as those that compose equation (5), example 2, before they were combined into two *terms.*)

Transposing the unknown term to the first member, the known to the second member, and solving,

$$41y = -123,$$

$$y = -3. \qquad (7)$$

Substitute in (3),

$$x = \frac{-15 + 31}{8}$$

$$= 2.$$

Check as before.

EXERCISE 17

Solve by comparison, verifying each result:

1. $3x + 6y = 7,$
   $4x - 2y = 1.$

2. $5y + 4x = 6,$
   $2y - x = 1.$

3. $8x + 15y = -8,$
   $43y - 12x = 12.$

4. $15x - 17y = 16,$
   $13x + 11y = 1.$

5. $6x + 3y = \frac{3}{2},$
   $5x - 3y = 4.$

6. $3x + 3y = 2,$
   $9x - 15y = -2.$

49. In stating written problems it is often convenient to use two or more unknown numbers rather than form a single equation in one unknown as was done in exercise 8. Care must be taken to form as many equations as there are unknown numbers.

Ex. A number is composed of two digits. One half the number is equal to twice the sum of the digits, and if 18 is added to the number the digits of the resulting number will be those of the original number written in reverse order. Find the number.

Let t = the digit in tens' place,
and u = the digit in units' place.
Then, $10t + u$ = the number. (Exercise 2, example 14.)

By the conditions,

$$\frac{10t + u}{2} = 2(t + u), \qquad (1)$$

$$10t + u + 18 = 10u + t,$$

or

$$9t - 9u = -18,$$

$$t - u = -2. \qquad (2)$$

From (1) $6t - 3u = 0,$

or $2t - u = 0.$ (3)

Subtracting (2) from (3) $t = 2.$ (4)

Substituting the value of t found in (4) in (2)

$$2 - u = -2,$$

and $u = 4.$

Whence, $10t + u = 24$, the required number.

EXERCISE 18

1. Find two numbers such that the sum of 3 times the first and 5 times the second is 24, and the sum of $\frac{1}{3}$ of the first and $\frac{1}{2}$ of the second is 2.

2. The altitude of a trapezoid is 8′ and its area 48 square feet. If the lower base is increased 4′, its area will be increased 16 square feet. Find its upper and lower bases. (Any trouble here?)

3. A cubic foot of steel and a cubic foot of water together weigh $552\frac{1}{2}$ pounds. The difference between their weights is $427\frac{1}{2}$ pounds. How many times heavier than water is steel? This quotient is called the specific gravity of steel. (Remember these results.)

4. The perimeter of a rectangular field is 240 rods, and its length is 40 rods more than its breadth. Find its area.

5. A field of wheat is 80 rods longer than it is wide. The farmer uses a combined harvester and thresher that cuts an 11-foot swath. After making 15 rounds, the indicator shows that $27\frac{1}{2}$ acres have been cut and have yielded 660 bushels. Find the yield per acre and the dimensions of the field.

6. Find two numbers such that n times the first added to k times the second is c, and k times the first minus n times the second is b.

7. The difference between the bases of a trapezoid is 6, the area 120, and the sum of the bases 30. Find the dimensions. *Can you construct* the trapezoid?

50. When a problem involves three variables the method of solution is similar to that in two variables. Care must be taken to eliminate the *same* unknown in each equation.

In solving simultaneous equations it is necessary that there be as many equations as there are variables in the problem.

Ex.
$$2x - y + 5z = 15, \qquad (1)$$
$$3x + 2y - 3z = -2, \qquad (2)$$
$$4x - 2y + 7z = 21. \qquad (3)$$

Multiply (1) by 2 and add the result to (2),
$$7x + 7z = 28. \qquad (4)$$
Add (2) to (3), $$7x + 4z = 19. \qquad (5)$$
We now have two equations in x and z.

Subtract (5) from (4), $$3z = 9,$$
$$z = 3. \qquad (6)$$
Substitute value of z in (5),
$$7x + 12 = 19,$$
or $$x = 1. \qquad (7)$$
Substitute (7) and (6) in (1),
$$2 - y + 15 = 15, \qquad (1)$$
or $$y = 2.$$
Check as before, substituting these values in (2) and (3).

EXERCISE 19

Solve the following:

1. $2x + 3y - 4z = 12,$
$x - 2y + 5z = -6,$
$3x + 2y - 6z = 13.$

2. $5y - 3x + 2z = 5,$
$5z - 3y + 2x = -9,$
$5x - 3z + 2y = 12.$

3. $3x - 2y = 5,$
$3y - 2z = 25,$
$3z - 2x = -25.$

4. $\frac{1}{2}x - \frac{1}{4}y + \frac{3}{2}z = \frac{25}{4},$
$\frac{1}{5}x + \frac{2}{5}y - \frac{3}{5}z = 0,$
$\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z = -\frac{5}{3}.$

5. $\frac{1}{4}x + \frac{1}{3}y + \frac{1}{2}z = \frac{13}{12},$
$\frac{1}{3}x + \frac{1}{2}y + \frac{1}{4}z = \frac{13}{12},$
$\frac{1}{2}x + \frac{1}{4}y + \frac{1}{3}z = \frac{13}{12}.$

6. An examination contained 15 problems. Each pupil received 7 credits for a problem solved, and 3 demerits for a problem missed. One boy had 45 marks more to his credit than he had against him. How many problems did he miss?

7. Find 3 numbers such that if 3 times the first be added to one half the sum of the second and third, the sum will be 137. If half the sum of the second and third be subtracted from the first the difference will be 23, the sum of the three numbers being 74.

8. If the order of the digits of a certain three-place number be inverted the sum of this number and the original number is 444, and their difference is 198. The first digit is equal to the sum of the other two. Find the number.

9. A merchant buys lemons, some at 4 for 5 cents, others at 3 for 4 cents, spending $2.80. Again he buys as many of the first kind as he had bought of the second kind, and as many of the second kind as he had bought of the first, this time spending $2.78. How many of each kind were purchased?

Solution of Inequalities

51. In solving inequalities the rules for solving equations hold except in some instances where change of sign takes place (§§ 37–41).

EXERCISE 20

If x be positive, which fraction has the greater value?

1. $\frac{x+5}{4}$ or $\frac{x+7}{6}$.

2. $\frac{2x+9}{6}$ or $\frac{2x+5}{2}$.

(Test 1 and 2 by substituting values for x.)

Find the limits of x for which these are true, and show that your results are correct:

3. $3x+5<x+2$.*

4. $5x-2>2x+7$.

5. $6x-4 \quad <0$.

6. $\frac{7x+14}{9}>0$.

* Solve as you would an equation.

Find the limits of the values of x which satisfy the following inequalities simultaneously (verify your results):

7. $$2x+6>0. \quad (1)$$
$$-x+5>0. \quad (2)$$

From (1), $x>-3. \quad (3)$

From (2), $x-5<0$ (§ 40),

and $x<5.$

Then, to satisfy the conditions of (1) and (2), x must lie between -3 and $+5$.

8. $24-3x>0,$
$16+2x>0.$

9. $5x+10>0,$
$3x-9<0.$

10. $4x-7<0,$
$6x-2x>0.$

11. $5x+2>0,$
$2x-5x>0.$

What values of x and y satisfy the following (§ 41):

12. $2x+3y>10,$
$3x-2y=8.$

13. Seven times the number of boys in the algebra class less 45 is greater than five times their number added to 13, and six times the number of boys less 22 is less than four times the number of boys added to 40. How many boys in the class?

14. Find the digit such that twice the digit increased by 3 is greater than one fifth the digit increased by 11.

15. Find a multiple of 18 such that five times the number decreased by 220 is greater than three times the number decreased by 42. Is there more than one such multiple?

16. Find a multiple of 16 such that one half of it decreased by 28 is less than one third of it increased by 16. Does more than one multiple satisfy this condition?

17. Find a line containing an integral number of inches such that five times the length of it decreased by 165 is greater than three times its length increased by 14. Is there more than one such line?

Supplemental Applied Mathematics

1. The 20th Century Limited of the Lake Shore Railroad runs 54.5 miles per hour. Find the speed in feet per second.

2. There are 24 gas stoves in the domestic science kitchen. If each stove burns 2.32 cubic inches of gas per second, find the cost of gas for one class of pupils during a period of 80 minutes. Allow 15 minutes waste time (when stoves are not in use). (Gas at 75¢ per thousand cubic feet.)

3. The power of running a motor in the pattern shop is .044 Kilowatt per hour. During a period of 80 minutes, 24 motors are running on an average of 60 minutes. Find the amount of power used. Find the cost at $0.0635 per Kilowatt hour.

4. The Southern Talc Company is able to ship freight in carload lots, by the long ton, and charge its customers the cost price per ton F. O. B. North Carolina, and base its charges on the short ton. Find the Talc Company's profit on 10 cars of 34 tons each, shipped from North Carolina to Chicago. Freight rate $3.50 per ton.

5. To find the weight of timber in the rough: multiply length in feet by breadth in feet by thickness in inches and multiply that product by one of the following factors. For Oak, 4.04; Elm, 3.05; Yellow Pine, 3.44; White Pine, 2.97. The result is in pounds. Measure six pieces of lumber in the shops and find the weight of each piece. Make accurate drawings of each piece.

6. The weight of a grindstone is found by multiplying the square of the diameter in inches by thickness in inches and this product by .06363.

$(\text{Diameter})^2 (\text{Thickness}) (.06363) =$ weight in pounds. (Remember this formula.) Find weight of grindstone in your shop. Make diagram.

7. The weight of one cubic foot of water is $62\frac{1}{2}$ pounds. *Find the weight of* one gallon.

8. Stock must be ordered for making 10,000 steel pins, each $\frac{1}{2}''$ diameter by $2''$ long. The rods for making these pins are $\frac{1}{2}''$ in diameter and $12'$–$0''$ long. Allow $\frac{1}{16}''$ waste in cutting each pin. How many rods will be needed? What will they weigh?

9. For flavoring, 1 ounce of chocolate is equivalent to 2 tablespoonfuls of cocoa. Chocolate costs 22 cents a cake (8 ounces), while cocoa costs 25 cents per box ($\frac{1}{2}$ pound). There are 2 cups of cocoa in one box. Which is cheaper to use?

10. Chocolate contains 48.7% fat, and cocoa, 28.9%. If cocoa were used in place of 2 ounces of chocolate, how many tablespoonfuls of butter would need to be added to make the same quantity of fat as in chocolate, butter containing 85% fat, and one tablespoonful weighing $\frac{1}{2}$ ounce?

11. Butter contains 85% fat, cream 18.5% fat, milk 4% fat. Calculate the quantity of butter necessary to use with 2 cups of milk to produce the same quantity of fat as 2 tablespoonfuls of butter and 2 cups of cream, one pint of cream weighing one pound, and one pint of milk one pound and one ounce.

12. Potatoes pared and then boiled lose 2.7% starch; potatoes boiled with the skins on lose .2% starch. Find difference in quantity of starch lost by cooking 6 potatoes, one potato weighing $4\frac{1}{4}$ ounces.

13. Boiled Irish potatoes contain 20.9% carbohydrates. Cooked sweet potatoes contain 42% carbohydrates. An Irish potato weighs $4\frac{1}{4}$ ounces, and a sweet potato 6 ounces. Find difference in quantity of carbohydrates in six of each kind of potatoes.

14. Find the reciprocal of 5280. Would you rather divide a number by 5280 or multiply the same number by its reciprocal? Which would give the larger result? Why?

15. Multiply 18649 cubic inches by the reciprocal of 1728. Is the result cubic feet?

16. Find the weight of a grindstone $36''$ in diameter and $4\frac{1}{2}''$ thick. What will it weigh after it has been worn down $2\frac{1}{4}''$?

17. A class of 15 girls wish materials and thread for cooking aprons. Each apron requires $2\frac{1}{2}$ yards muslin at 12¢ per yard, and a spool of thread at 5¢. How much material and thread for the class?

18. How much money was spent in making the purchases in example 17? How much does each girl pay for her share?

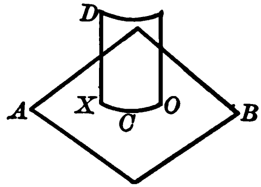
19. If a girl earned $22\frac{1}{2}$¢ a piece making these aprons, how many would she have to make to earn $5.50 a week?

20. If these aprons sold at 85¢ each, what would be the profit per apron?

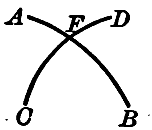
CHAPTER V

Lines, Angles, Triangles

52. A portion of space is called a **geometric solid.** The boundary between the solid and the remaining space is called the **surface of the solid.** For example, if the sides, bottom, and top of a box were considered to have no thickness, they might be called the surface of the solid.



53. If two surfaces intersect, their intersection is called a **line.** Thus, if surfaces AB and DO intersect in XCO, their intersection, XCO, is called a **line.**



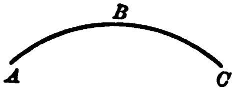
54. An intersection of two lines is called a **point.** Thus, F, the intersection of lines AB and CD, is a point.

55. A line may also be regarded as the path of a moving point.

56. The line is said to be **straight** when it does not change its direction at any point.

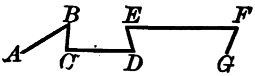
A line has no limit in its length. When a part of it is indicated, we speak of it as a segment or sect of the line. Thus, A ―――――――― B, AB indicates a straight line passing through points A and B. The part of the line between A and B is a segment or sect of the line. A line may also be indicated by a small letter placed somewhere on the line.

E.g., ――――a―――― is read "the line a."



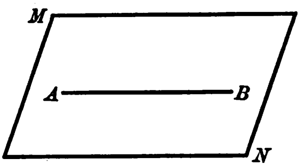
57. A **curved line** changes its direction at every point. *E.g.*, the curve *ABC*.

58. A **broken line** is composed of segments of successive straight lines which have different directions and which, pair by pair, have a point in common. *E.g.*, *ABCDEFG* is a broken line.



59. The word *line*, unmodified, usually means a straight line.

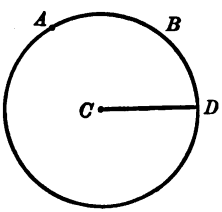
60. A **plane** is a surface such that if any two of its points be joined by a straight line, the line will lie entirely in the surface.



Thus, in plane *MN*, if any two points, *A* and *B*, are joined by a line, the line will lie entirely within the surface.

61. **Plane Geometry** treats of combinations of points and lines all lying in the same plane.

62. A **circle** is a portion of a plane bounded by a curved line every point of which is equidistant from a point within called the center.



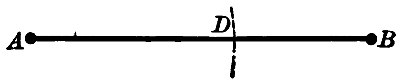
The curved line is the **circumference.**

The distance from the center to any point in the circumference is the **radius,** as *CD*.

An arc is any part of the circumference, as *AB*.

PROBLEM

63. *To draw a straight line equal to a given straight line.*



Given a line MN.

To draw a straight line equal to MN.

1. Draw an indefinite line AB.

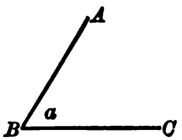
2. With A as a center and MN as a radius, describe an arc cutting AB at D.

3. AD is the required line.

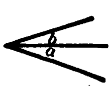
64. An **angle** is the figure formed by two lines drawn from a point.

The point is called the **vertex**.

The above angle is read ABC, the letter at the vertex always being between the other two. The angle may also be designated by a small letter placed between the sides of the angle, as the angle x. This may be written $\angle x$.* The size of an angle does not depend upon the length of the sides, but upon the difference in the direction of the sides.



65. Two angles are said to be **adjacent** when they have a common vertex and a common side between them.



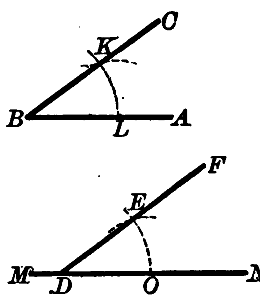
Thus $\angle a$ and $\angle b$ are adjacent.

66. Two angles are equal when one can be so applied to the other that they will coincide throughout.

* Also when only one angle is formed at a vertex, it may be read by a single letter, as $\angle B$, read "angle at B." *Small letters* denote values; capital letters denote *position only*.

Problem

67. *To construct an angle equal to a given angle.*



Given $\angle ABC$.

To construct an angle equal to it.

1. Draw any line MN.
2. Take any point D on MN for the vertex of the angle.
3. With B as center and any radius BL, describe an arc cutting AB and BC at L and K, respectively.
4. With BL as radius and D as center, describe an arc cutting DN at O.
5. With O as center and distance LK as radius, describe an arc cutting the former arc at E.
6. Draw DEF.
7. Then NDF is the required angle.

68. A **theorem** is a truth requiring proof.

In proving theorems the following axioms, in addition to those in § 23, are useful:

Ax. 9. But one straight line can be drawn between two points.

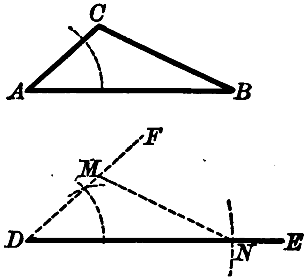
Ax. 10. A straight line is the shortest distance between two points.

69. A **triangle** is a portion of a plane bounded by three straight lines. The lines are called the sides of the triangle. The points where the sides meet are the **vertices**.

Three-sided figures play an important part in the study of geometry and in its applications to many problems in mathematics. Much of the work in measurement, in surveying, and in other kinds of engineering depends upon the triangle.

THEOREM I

70. *Two triangles are equal when two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*



Draw a $\triangle ABC$.*

On any line DE take DN equal to AB (§ 63).

At D construct $\angle D = \angle A$ (§ 67).

On DF take $DM = AC$ and draw MN.

In the above theorem, we have three equations *given*. (This given part is called the **hypothesis.**)

The **given** equations are:

$$\left.\begin{aligned} AC &= DM \\ AB &= DN \\ \angle A &= \angle D \end{aligned}\right\}$$

(Don't forget that *what is given, the parts that are given,* are the tools you have to work with.)

To prove the geometric equation,

$\triangle ABC = \triangle DNM$. (This is called the conclusion.)

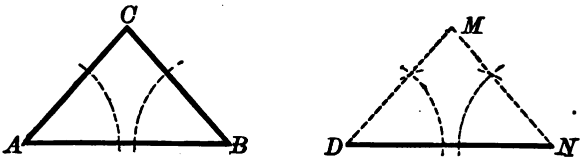
Proof. 1. Place $\triangle DNM$ on $\triangle ABC$ in such a manner that DN will coincide with its equal AB, D falling on A.

2. DM will take the direction AC because $\angle D = \angle A$.
3. Point M will fall on point C since $DM = AC$.
4. Hence line MN will coincide with line CB (Ax. 9).
5. Then $\triangle ABC = \triangle DNM$, since they coincide throughout.

* In reading the letters at the vertices of a triangle always read counter-clockwise, beginning as far as is practicable at the lower left-hand corner, e.g., ABC, DNM in above triangles.

THEOREM II

71. *Two triangles are equal when a side and two adjacent angles of the one are equal, respectively, to a side and two adjacent angles of the other.*



What is the hypothesis? What is the conclusion?
Draw a $\triangle ABC$.
Draw a line DN equal to AB.
At D construct an angle equal to $\angle A$.
At N construct an angle equal to $\angle B$.
Extend the sides of these angles to meet at M. We have
Given in the $\triangle$s ABC and DNM

$$DN = AB,$$
$$\angle D = \angle A,$$
$$\angle N = \angle B.$$

To prove $\triangle ABC = \triangle DNM$.

Proof. 1. Place $\triangle DNM$ upon $\triangle ABC$ in such a way that DN will coincide with its equal AB, point D falling on A.

2. Since $\angle D = \angle A$, side DM will take the direction AC and M will fall somewhere on AC.

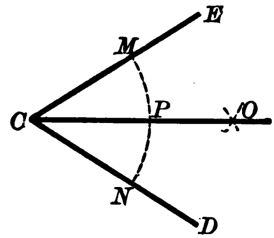
3. Since $\angle N = \angle B$, side NM will take the direction BC and M will fall somewhere on BC.

4. Since M falls on both AC and BC, it must fall at C, their point of intersection.

5. Hence $\triangle$s ABC and DNM coincide throughout and are *equal.*

PROBLEM

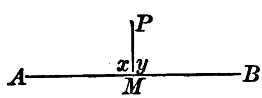
72. *To bisect a given angle.*



1. Draw an angle DCE.
2. With C as center and any radius CM, describe an arc cutting CE and CD at M and N, respectively.
3. With M and N as centers and a radius greater than one half MN, describe arcs intersecting at O.
4. Draw CO.
5. Then CO is the bisector of $\angle DCE$.

73. The right angle. If one straight line meets another straight line in such a manner that the adjacent angles (§ 65) formed are equal, the angles are called right angles and the lines are **perpendicular** to each other.

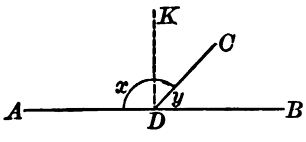
Thus, if AB is a straight line and $\angle x = \angle y$, x and y are each right angles, and MP is perpendicular to AB.



($\perp$ is the abbreviation for "perpendicular to.")

74. If one straight line meets another straight line, adjacent angles are formed. If these angles are not right angles, the one greater than a right angle is called **obtuse**. The one less than a right angle is called **acute**. Either angle is said to be ***oblique***.

75. If a line CD meets another line AB, two adjacent angles x and y, are formed. Suppose y less than x. If CD is made to rotate toward A, about D as a pivot, $\angle x$ diminishes and $\angle y$ increases. It is evident that at some position DK of DC, $\angle x$ will equal $\angle y$. There can be only one such position. *Then, one and only one perpendicular can be erected to a given line at a given point in the line.*



76. It follows from § 75 that:

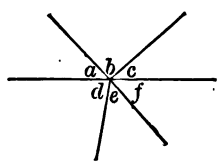
(*a*) *Two right angles are equal.*

(*b*) *If one straight line meets another straight line, the sum of the adjacent angles formed is equal to two right angles.* That is, in the figure of § 75.

$$\angle x + \angle y = \angle ADK + \angle KDB = 2 \text{ rt. } \measuredangle s.$$

(*c*) *The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.*

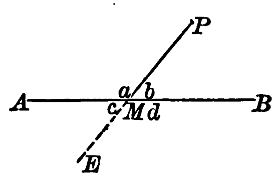
(*d*) *The sum of all the angles about any point in a plane is equal to four right angles.*



$$a + b + c + d + e + f = 4 \text{ rt. } \measuredangle s.$$

(*e*) *If the sum of two adjacent angles is two right angles, their exterior sides lie in the same straight line.*

Let $a + b = 2$ rt. $\measuredangle s$.



Extend PM to E. If AM and MB were not in a straight line, the sum of c and d would not be two right angles, and this is contrary to (*d*).

77. Species of triangles.

The right triangle is a triangle one of whose angles is 90°

or a right angle. The side opposite the right angle is called the hypotenuse. The other two sides are called the legs.

All triangles except right triangles are **oblique triangles.**

If an oblique triangle contains one obtuse angle, it is called an **obtuse angled triangle.**

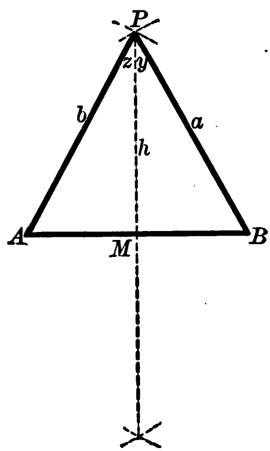
An **acute angled triangle** has each angle less than a right angle.

Triangles are also named according to their lengths of sides.

A **scalene triangle** has no two sides equal. An **isosceles triangle** has two sides equal. An **equilateral triangle** has three sides equal.

THEOREM III

78. *In an isosceles triangle the angles opposite the equal sides are equal.*



Draw line AB. With A and B as centers and the same radius greater than $\frac{1}{2}$ AB, describe arcs intersecting at P. Draw PA and PB. We now have the isosceles triangle ABP, with $b = a$.

What is **given** ?

To prove $\angle B = \angle A$.

Proof. 1. Draw line PM bisecting APB (§ 72).

2. In $\triangle$ AMP and MBP,

$$b = a;\ h = h;\ \angle z = \angle y.$$

3. Hence, $\triangle AMP = \triangle MBP$ (§ 70).

4. *Then* $\angle B = \angle A$, being like parts of equal figures.

79. In equal figures, corresponding lines or angles are called homologous. It follows that in equal figures homologous parts are equal.

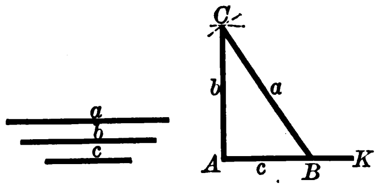
NOTE. *In equal triangles equal sides lie opposite equal angles.* Thus, in equal $\triangle$ AMP and MBP, § 78, side $a =$ side b, then $\angle A$ is homologous to $\angle B$.

PROBLEM

80. *To construct a triangle when three sides are given.*

Let the given sides be a, b, c.

1. Draw an indefinite line AK.
2. On AK take $AB = c$.
3. With A as center and b as radius, describe an arc.
4. With B as center and a as radius, describe an arc intersecting with the former arc at C.
5. Draw AC and CB.
6. Then ABC is the required triangle.



THEOREM IV

81. *Two triangles are equal when three sides of one are equal, respectively, to three sides of the other.*

Draw a triangle ABC.

Construct a second triangle DEF whose sides are equal to a, b, c, respectively (§ 80). Namely, $d = a$, $e = b$, $f = c$.

What is **given**?

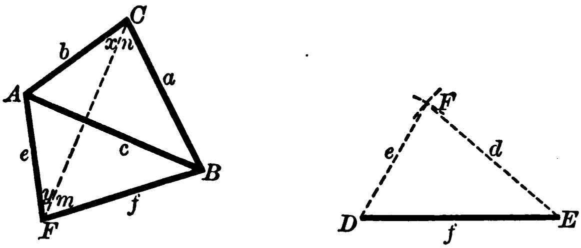
To prove $\triangle DEF = \triangle ABC$.

Proof. 1. Apply $\triangle$ DEF to $\triangle$ ABC in such a way that f will coincide with its equal c, and F will fall opposite C.

2. Draw CF.

3. In $\triangle AFC$, $b = e$. (By Hypothesis.)
4. Then $\angle x = \angle y$ (§ 78).

(In an isosceles triangle the angles opposite the equal sides are equal.



5. Similarly, $\triangle FBC$ is isosceles and $\angle n = \angle m$.
6. Adding the equations in steps 4 and 5,

$$\angle x + \angle n = \angle y + \angle m.$$

7. Then, in $\triangle$s ABC and AFB,

$$b = e$$
$$a = f$$
$$\angle ACB, \text{ or } \angle x + \angle n = \angle AFB \text{ or } \angle y + \angle m.$$

8. Hence, $\triangle ACB = \triangle AFB$ (§ 70).

(Two triangles are equal when two sides and the included angle of th one are equal, respectively, to two sides and the included angle of th other.)

9. But $\triangle AFB$ is the same as $\triangle DEF$.
10. Hence, $\triangle DEF = \triangle ABC$.

NOTE. The **base** of a triangle is the side on which it is sup posed to rest. Any side may be considered the base. Th **altitude** is a perpendicular (§ 73) from any vertex to the oppo site side. Theorems are truths to be used. Remember eacl one. Notice that theorems I and III are both used in provin *theorem IV.*

EXERCISE 21

1. Construct a triangle each of whose sides is 8″.

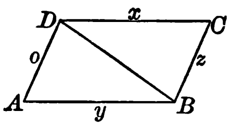
2. The sides of a triangle are 3″, 4″, 5″, respectively. Construct the triangle.

3. The sides of a triangle are 13′, 15′, 16′, respectively. Draw this triangle to a scale, using $\frac{1}{2}'' = 1' - 0''$.

4. Draw three altitudes of the triangle constructed in example 3.

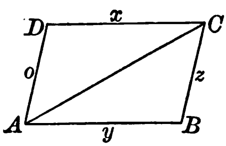
5. Construct a triangle whose sides are 15′, 16′, 28′. Draw the three altitudes.

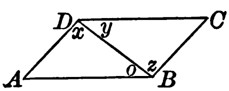
6. Measure the three altitudes of the triangle constructed in example 1. How do they compare? Is this comparison the same in altitudes of other triangles of this exercise?



7. In the figure *ABCD*, in ⚠ *ABD* and *CDB*, $x = y$ and $o = z$. Compare the triangles. Give reasons for your conclusion.

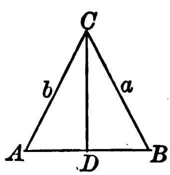
8. In the figure *ABCD*, in triangles *ABC* and *CDA*, *x* is equal to *y*, *0* is equal to *z*, $\angle D$ is equal to $\angle B$. Compare the triangles. Give reason for your conclusion.





9. In triangles *ABD* and *CDB*, $\angle x = \angle z$, and $\angle y = \angle o$. Compare the triangles. Why?

10. If in triangle *ABC*, $b = a$, and *CD* is so drawn that $AD = DB$, compare triangle *ADC* with triangle *CDB*.



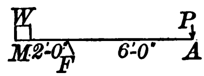
11. On a six-inch base construct a triangle whose remaining sides are each 14″. What do you know about this triangle?

12. The sum of two numbers is 24. One number is four more than the other. What are the numbers? (See example 10, exercise 8.)

13. The difference of two numbers is 8, and their product is 160 more than the square of the less number. Find the numbers.

14. The sides of a right triangle (§ 77) are as 4 to 3.* The square of the hypotenuse is equal to the sum of the squares of the other two sides. The hypotenuse is 25. Find the sides and the area.

15. In a lever AM, $P =$ power; $F =$ fulcrum, on which the lever rests; $W =$ weight. Note that if the lever is pressed downward at P, W tends to rise. It has been found that: *power* times the *distance*, from where the power is applied, to the fulcrum is equal to the *weight* times the *distance* from the weight to the fulcrum, or



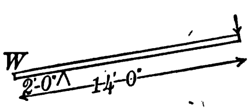
$$P \cdot AF = W \cdot MF.$$

AF is called the *power* or *force arm*, MF, the *weight arm*.

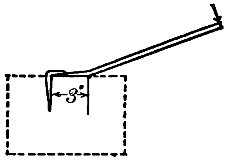
If the power arm is $6' - 0''$, the weight arm $2' - 0''$, and the weight 200 pounds, what power is necessary to lift the weight?

Note. In these problems, the weight of the lever is neglected.

16. With a $4'' \times 4''$, 14 feet long, used as a lever, a 100-pound boy on the end of the power arm is just able to lift a weight W, when the fulcrum is $2' - 0''$ from the weight. What weight does he lift?

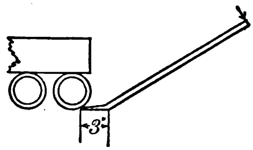


* In such statements fractions are avoided by letting the required values be represented by a multiple of the unknown number. For example: Let 3s be *one side, and* 4s *the* other.

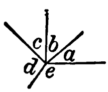


17. A crowbar is 5′—0″ long. The fulcrum is 3″ from one end. A force of 20 pounds at the other end of the bar is required to pull a railroad spike from a tie. With what force does the spike hold?

18. With the crowbar and fulcrum as in example 17, two boys weighing 100 lb. and 115 lb., respectively, were able to move a freight car standing on a siding. What force was necessary to start the car?



19. The length and breadth of a rectangular tank are as 4 to 3. The water in the tank is frozen to a depth equal to one fourth the width of the tank. The area of the bottom of the tank is 48 square feet. Find the weight of the ice. (Ice weighs $57\frac{1}{2}$ pounds per cubic foot.) Which is heavier, water or ice? Why is this necessary? See exercise 18, example 3.

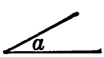


20. Five lines are drawn from a point forming angles *a, b, c, d, e*. The sum of *a* and *b* is 90°; *d* is equal to 12° more than *a*; *c* is equal to *b*; *e* is equal to twice *d*. Find the angles. (See § 76, *d*.) Is the drawing correct?

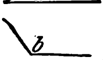
21. Divide 48 into three parts, such that the first part shall be twice the second, and the third 8 more than the first.

22. A rectangle and a square have the same altitude. The base of the rectangle is 8′ more than the base of the square. The areas differ by 64 square feet. Find the dimensions of each.

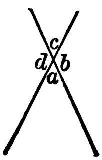
82. An **acute angle** is an angle that is less than a right angle. Ex. Angle *a*.



An **obtuse angle** is an angle that is greater than a right angle and less than two right angles. Ex. *Angle b.*



83. When two straight lines intersect, the opposite angles formed are called **vertical angles.**



Thus, a and c are vertical angles, also b and d.

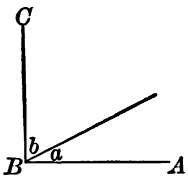
84. An angle is measured by comparing it with another angle considered as the unit of measure.

The most general unit of measure is $\frac{1}{90}$ of a right angle, and is called a **degree.**

The degree is divided into 60 equal parts called **minutes,** and the minute into 60 equal parts called **seconds.**

The abbreviations for degrees, minutes, seconds, are $^\circ$, $'$, $''$, respectively.

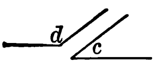
85. The right angle (§ 73) is measured by one fourth of the circumference of a circle, or 90°.



86. If the sum of two angles is a right angle, each is a **complement** of the other, and the angles are said to be **complementary.**

Thus, if ABC is 90°, a is the complement of b, their sum being 90°.

87. If the sum of two angles is 180° or two right angles, each is the **supplement** of the other, and the angles are said to be **supplementary.**



Thus, if the sum of angles c and d is 180°, c and d are supplementary and each is the supplement of the other.

88. It follows from § 87, that if two supplementary angles, as c and d in the figure are adjacent (§ 65), their exterior sides lie in the same straight line.

Then c and d are **supplementary adjacent angles.**

EXERCISE 22

1. Find the complement of 38°; the supplement of 38°.
2. Are 36° 30′ and 53° 30′ complementary? Why?

3. Are $130° 19''$ and $49° 41'$ supplementary? Why?

4. Find the complement of each of these angles: $24°$; $36°$; $72°$; $60°$; $30°$; $45°$; $30° 15'$; $24° 15' 20''$; $47° 10.5'$; $b°$.

5. Find the supplement of each of these angles: $22°$; $96°$; $120°$; $150°$; $24° 8'$; $27° 19' 36''$; $45° 4.4'$; $90°$; $b°$.

6. Find the complement of the supplement of: $120°$; $130°$; $135°$; $c°$. Does this last result give a simple formula?

7. Find the supplement of the complement of: $3°$; $24°$; $28° 5' 30''$; $45°$; $90°$; $c°$. Does this last result afford a means of simplifying other parts of this example? Illustrate.

8. Angle at $A = 36°$, $\angle B = 74°$. Find the supplement of their sum.

9. Angle at $A = 29°$, $\angle B = 31°$. Find the complement of their sum.

10. Angle at $A = 75°$, $\angle B = 15°$. Find the complement of their difference.

THEOREM V

89. *Any side of a triangle is greater than the difference of the other two sides.*

Draw any triangle ABC, the sides opposite A, B, C, being a, b, c, respectively.

We have

Given a any side of $\triangle ABC$, and $c > b$.

To prove $a > c - b$.

Proof. 1. $a + b > c$.

A straight line is the shortest distance between two points. (§ 68, Ax. 10).

2. $a > c - b$. (Transposing b, § 42.)

3. Hence, any side, a, is greater than the difference of the *other two sides.*

THEOREM VI

90. *The sum of two sides of a triangle is greater than the sum of two lines drawn from any point within the triangle to the extremities of the third side of the triangle.*

Draw* any triangle ABC.

From any point P, within the triangle draw PB and PC.

Call side AB, c; AC, b; PB, e; PC, d.

We now have:

Given $\triangle ABC$, with lines d and e drawn from any point P within the triangle.

To prove $c + b > d + e.$

Proof. 1. Produce BP to E.†

Call PE, o; AE, x; EC, y.

2. $c + x > e + o.$

A straight line is the shortest distance between two points (§ 68, Ax. 10).

3. $o + y > d$ (§ 68, Ax. 10).
4. Add 2 and 3,

$$c + x + o + y > d + o + e.$$

5. Subtract o from each side of this inequality,

$$c + x + y > d + e \text{ (§ 39).}$$

6. But $x + y = b$ (§ 23, Ax. 6).
7. Whence, substituting for $x + y$ its equal b,

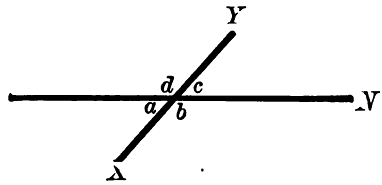
$$c + b > d + e.$$

*** This description of the drawing is not a part of the demonstration. Your demonstration depends solely on the three paragraphs: *Given, To prove, Proof.* Be sure you know what you have *given* you to work with and what you wish *to prove.***

† Remember that any additional lines you draw in a figure must be dotted *lines.*

THEOREM VII

91. *If two straight lines intersect, the vertical angles are equal.*



Draw two intersecting lines, MN and XY, forming the angles a, b, c, d. We then have

Given two intersecting lines, MN and XY, and vertical angles a, c, and b, d.

To prove $a = c$ and $b = d$.

Proof. 1. $a + b = 2$ rt. $\angle s$ (§ 88).

2. $b + c = 2$ rt. $\angle s$ (§ 88).

3. Subtract 2 from 1 (§ 23, Ax. 2),

$$a - c = 0,$$

or

$$a = c.$$

4. The pupil may prove $b = d$.

What is the hypothesis in the above theorem? What is the conclusion?

PROBLEM

92. *To draw a perpendicular bisector of a straight line.*

Draw line AB.

To draw a perpendicular bisector of AB,

1. With A as center and any radius greater than one half AB, describe arcs on each side of line AB.

2. With B as center and the same radius, describe arcs intersecting the arcs already drawn at C and at D.

3. Draw CD.

4. Then CD is the perpendicular bisector of AB.

The reason for this construction will be given in § 95. F, the point of intersection of CD and AB, is called the foot of the perpendicular.

Note that C and D are each equally distant from A and B.

Problem

93. *To draw a perpendicular to a line at any point in the line.*

1. Draw a line AB.
2. Take any point P in the line AB.
3. To erect a perpendicular to line AB at P.
4. With P as center and any radius, describe arcs intersecting AB, or AB produced, at C and E.
5. With C and E as centers and radius greater than one half CE, describe arcs intersecting on one side of AB.
6. The pupil may show that the perpendicular may now be drawn and that this problem is an application of § 92.

Why must the radius be greater than one half CE?

EXERCISE 23

1. How many points determine a line? What do you mean by determine as used in this sense?

2. How many conditions can be imposed on a line? What properties besides length has a straight line?

Theorem VIII

94. *If a perpendicular is erected at the middle point of a line,*

I. *Any point in the perpendicular is equidistant from the extremities of the line.*

II. *Any point not in the perpendicular is unequally distant from the extremities of the line.*

Draw line $KF \perp$ line AB at its middle point F.

From P, any point in KF, draw lines PA and PB.

We then have,

I. **Given** two lines PA (x) and PB (y) drawn from any point P in the perpendicular to AB at its middle point.

To prove $\qquad x = y.$

Proof. 1. In triangles AFP and PFB,

$o = b$ (Constr.)
$h = h$ (Identical)
$\angle AFP = \angle BFP.$
(All rt. $\angle$s are equal.)

2. Then, $\triangle AFP = \triangle PFB.$

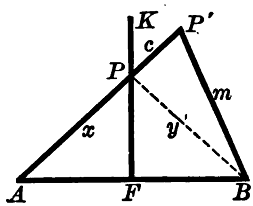
(Two triangles are equal when two sides, etc., § 70.)

3. Then, $x = y.$

(In equal figures corresponding parts are equal.)

Does this remind you of Theorem III?

II. **Given** P' any point not in KF.



To prove $P'A \neq P'B,$*

or $x + c \neq m.$

Proof. 1. If P' is not in KF, $P'A$ or $P'B$ must intersect KF.

Suppose $P'A$ intersects KF at P.

Draw PB (y).

2. Then, $y + c \neq m$ (§ 68, Ax. 10).
3. But $y = x$ (§ 94, I).
4. Then, substituting x for its equal y in 2,

$$x + c \neq m.$$

95. From the result of § 94, we may conclude that:

I. A point equally distant from the extremities of a line lies in the perpendicular at the middle point of the line.

II. Two points each equally distant from the extremities of a line determine the perpendicular at the middle point of the line.

III. Two lines drawn from a point in the perpendicular to a line and cutting off equal distances from the foot of the perpendicular make equal angles with the perpendicular, and are *equal*. (The proof of III is left to the pupil.)

* $\neq$ is read "does not equal."

Problem

96. *To draw a perpendicular to a line from a given point without the line.*

Given line AB and point P without AB.

To draw $PF \perp AB$. Construction:

1. With P as center and a radius greater than the distance from P to line AB, describe an arc intersecting AB at E and D.
2. With E and D as centers and the same radius greater than one half ED, describe arcs intersecting at K.
3. Draw PK.
4. PK is the required perpendicular.

Note that P and K are each equally distant from E and D. Compare EF and FD (§ 94).

Theorem IX

97. *From a point without a line but one perpendicular can be drawn to the line.*

Draw line AB and $PF \perp AB$, also PH any other line meeting AB at H. We have

Given $PF \perp AB$, and PH any other line from P to AB.

To prove PH not perpendicular to AB.

Proof. 1. Produce PF to P' making $FP' = PF$.

2. Draw HP'.

3. Represent $\angle s$ PHF and $P'HF$ by x and y, respectively.

4. Then, $\angle x = \angle y$ (§ 95, III).

5. PFP' is a straight line (by construction).

6. Then, PHP' is not a straight line.

(But one straight line can be drawn between two points, Ax. 9.)

7. Then, $x + y \neq 180$. (§ 88)

8. Whence, $2x \neq 180$, and $x \neq 90$.

9. Hence, PH is not perpendicular to AB. And since PH is any line except PF, PF is the only perpendicular that can be drawn from P to line AB.

EXERCISE 24

1. The base of an isosceles triangle is 8″, the altitude 6″. Construct the triangle.

2. The altitude of an isosceles triangle is 8″, one leg is 10″. Construct the triangle.

3. The sides of a triangle are 5″, 12″, 13″, respectively. Construct the triangle.

4. The base of a triangle is 13″, the base angles each 45°. Construct the triangle. Measure the angle at the vertex.*

5. The sides of a triangle are 4″, 10″, 6″, respectively. Construct the triangle. Explain your result.

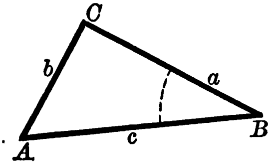
6. Two sides and the included angle of a triangle are 8″, 12″, 45°, respectively. Construct the triangle.

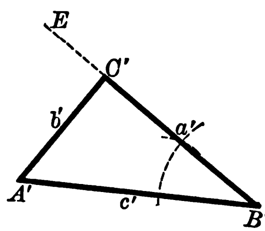
7. Draw two acute angles, a and b. At a given point on a line AB, construct an angle equal to the sum of these two angles.

8. Draw a triangle ABC, and at a given point P, on a line DE, construct an angle equal to the sum of the angles at A, B, and C.

THEOREM X

98. *Two right triangles are equal when the hypotenuse and an acute angle of the one are equal, respectively, to the hypotenuse and acute angle of the other.*





Draw right $\triangle ABC$, and side $A'B'$ (c') = side AB (c).

* *The vertex* opposite the base is *the vertex* of the triangle.

At B' construct $\angle B' = \angle B$, and draw $A'C' \perp B'E$. Represent BC, $B'C'$ by a and a', respectively, also AC, $A'C'$ by b and b', respectively.

We then have

Given rt. $\triangle$s ABC and $A'B'C'$ with $c = c'$, and $\angle B = \angle B'$.

To prove $\triangle ABC = \triangle A'B'C'$.

Proof. 1. Place $\triangle ABC$ on $\triangle A'B'C'$ in such a manner that c will coincide with its equal c'.

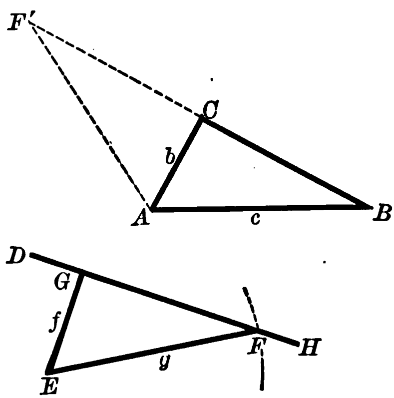
2. Since $\angle B = \angle B'$, a will fall on a'.

3. A falls on A', then side b will coincide with side b'. (But one $\perp$ can be drawn from a point to a line from a point without the line, § 97.)

4. Hence, the triangles are equal, since they coincide throughout.

Theorem XI

99. *Two right triangles are equal when the hypotenuse and a leg of the one are equal, respectively, to the hypotenuse and a leg of the other.*



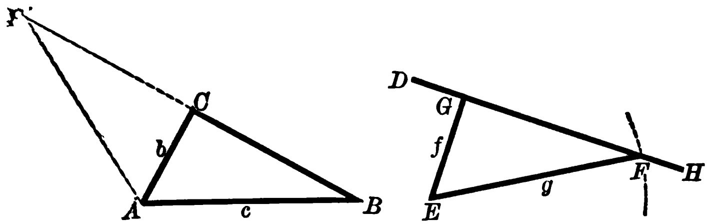
Draw rt. $\triangle ABC$. On line DH, at G, erect $EG \perp DH$ and equal to b. With E as center and c as radius, describe an arc intersecting DH at F. We then have

Given two right triangles, ABC and EFG, with $c = g$, and $b = f$.

To prove $\triangle ABC = \triangle EFG$.

Proof. 1. Apply $\triangle EFG$ to $\triangle ABC$ in such a manner that f *will coincide* with its equal b, vertex F falling at F'.

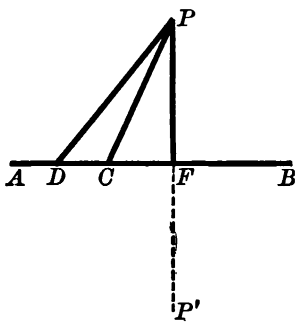
2. Since $\angle s$ ACB and ACF' are right angles, BCF' is a straight line (§ 76).



3. Since $c = g$ $\triangle BAF'$ is isosceles, and $\angle B = \angle F'$ (§ 78).
4. Then, $\triangle ABC = \triangle ACF'$ (§ 98).
5. Then, $\triangle ABC = \triangle EFG$. Why?

THEOREM XII

100. *If two unequal oblique lines drawn from a point in a perpendicular to a line, cut off unequal distances from the foot of the perpendicular, the more remote is the greater.*



Draw line AB, and $PF \perp AB$ at F, also lines PD and PC, DF being greater than CF.

To prove $PD > PC$.

Extend PF to P', making $FP' = PF$.
Draw DP' and CP'.
The *proof is left to the student* (§ 90).

THEOREM XIII

101. *If oblique lines are drawn from a point to a straight line and a perpendicular is drawn from the point to the line,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular.*

II. *The greater of two unequal oblique lines cuts off the greater distance from the foot of the perpendicular.*

I. Call AB the given line, P the given point, PF the perpendicular to AB, PE and PC the equal oblique lines.

The proof is left to the student.

(Prove $\triangle PFE = \triangle PFC$.)

II. Draw $PF \perp AB$, also lines PC (x) and PD (y), with $x > y$. We then have

Given $PF \perp AB$, and line $x >$ line y.

To prove $CF > DF$.

Proof. 1. If $CF = DF$, $x = y$ (§ 95, III). But this is contrary to hypothesis.

2. If $CF < DF$, $x < y$ (§ 100). This is also contrary to hypothesis.

3. Hence, if CF is not equal to or less than DF, it must be greater than DF.

102. The manner of proving § 101 is called "The Indirect Method." In § 101 what is the hypothesis? What the conclusion? Compare § 95, III. What is the relation between the hypothesis of § 95, III, and the conclusion of § 101, I? When the conclusion of one theorem is the hypothesis of the other, and the hypothesis of the one is the conclusion of the other, each theorem is the **converse** of the other.

EXERCISE 25

1. The line joining the vertex of an isosceles triangle to the *middle point of* the base bisects the vertical angle.

2. Two isosceles triangles have a common base but unequal altitudes. Show that the line connecting their vertices is perpendicular to the base and bisects the base.

3. If the altitude of a triangle bisects the base, the triangle is isosceles.

4. Two isosceles triangles are equal if a leg and the vertical angle of one are equal, respectively, to a leg and the vertical angle of the other.

5. The hypotenuse of each of two right triangles is $81' - 6''$. Each has an angle of $31° 41'$. Compare the triangles.

6. The hypotenuse and one leg of a triangle are 3 yards, 1 foot, and 1 yard, 2 feet, and 6 inches, respectively. The hypotenuse and one leg of another triangle are $10' - 0''$ and $5' - 6''$, respectively. Compare the triangles.

ORAL REVIEW

Give the results of the following:

1. $19 \cdot 17$.
2. $16 \cdot 15$.
3. $18 \cdot 16$.
4. $18 \cdot 5 - 6 \cdot 17$.
5. $24 \cdot 6 + 3 \cdot 17$.
6. $4 \cdot 18 - 3 \cdot 16$.
7. $24 \cdot 5 + 3 \cdot 17$.
8. $9 \cdot 8 + 5 \cdot 7$.
9. $6 \cdot 15 - 3 \cdot 14$.
10. $6 \cdot 11 - 7 \cdot 11$.
11. $9 \cdot 15 + 6 \cdot 15$.
12. $8 \cdot 17 + 7 \cdot 17$.
13. $8 \cdot 16 + 8 \cdot 16$.
14. $27 \cdot 3 - 4 \cdot 28$.
15. $10 \cdot 34 - 5 \cdot 17$.
16. $12 \cdot 8 + 12 \cdot 8$.
17. $-3 \cdot 15 - 3 \cdot 15$.
18. $-8 \cdot 16 + 7 \cdot 15$.

Supplemental Applied Mathematics

1. A grown person needs 3000 cu. ft. of air per hour that the functions of the body may be active. If a room 20 ft. by

15 ft. by 10 ft. were occupied by one person, how often would the air have to be completely changed to obtain pure air?

2. How often would the air in a room 12 ft. by 11 ft. by $8\frac{1}{2}$ ft. have to be completely changed to obtain pure air for two persons?

3. If there were 500 people in a lecture hall, how often would the air have to be completely changed to insure good ventilation? Nine square feet of floor space is allowed to each person, and the hall is 11 feet high.

4. For good ventilation the air in a room containing one person needs to be changed every 45 minutes. What is the approximate size of the room?

5. It has been found by experiment that the air in a room cannot be changed more than three times per hour without danger of drafts. What are the dimensions of a room that is just large enough to meet this ventilating requirement, when one person occupies the room?

6. It has been found by experiment that one gas jet, when burning, uses as much air as two persons. By changing the air in a room occupied by one person once an hour during the daytime, good ventilation is secured. How many times per hour does it need to be changed in the evening when one gas jet is burning?

7. When two persons occupy a room, good ventilation is secured in daytime by changing the air once each half hour. How often should the air be changed when 3 gas jets are lighted?

8. The air in a room occupied by one person needs to be changed once every hour in the daytime and three times every hour during the evening. How many gas jets are burning?

9. A kerosene lamp requires as much air as 4 persons. When 2 persons occupy a room, the air needs to be changed once every 2 hours. When a lamp burns in the room, how *often does the* air need to be changed?

10. In the daytime the air of a room containing 2 persons needs to be changed once every 45 minutes, and in the evening once every 15 minutes. The room is lighted by kerosene lamps. How many kerosene lamps are burning in the room?

11. The quantity of carbon dioxide given off by candles is about twice as much as that given off by gas. If the air in a room needs to be changed once every $1\frac{1}{2}$ hours when illuminated by gas, how often will it need to be changed when illuminated by candles?

12. For practical purposes architects figure 30 cubic feet of air per minute for each person. A classroom has a ventilating system. The room is $28' \times 23' \times 15'$ and contains 30 pupils. To insure good ventilation, how much air must be driven into the room and how many times per hour must the air be changed?

13. In hospitals it is customary to allow 50 cubic feet of air per minute per person. In a hospital ward $56' \times 9' \times 12'$ are 8 patients, a nurse, and 2 gas jets. How much air must be supplied per hour? How many times must it be changed per hour?

14. A train running 40 miles an hour strikes two torpedoes 400 feet apart. Sound travels 1090 feet per second. What time elapses between their reports at a station which the train is nearing?

15. According to some engineers, the sectional area of the cold-air box of a furnace should be equal to the combined areas of all the registers. There are 6 registers in a house, each 8 in. by 10 in. How large should the cold-air box be?

16. The cross section of a cold-air box is 2 ft. 3 in. by 1 ft. 8 in. There are 6 equal registers in the house. What is the area of each register?

17. The sectional area of a cold-air box is 495 sq. in. Each register measures 9 in. by 11 in. How many registers are there in the house?

18. According to other engineers, the sectional area of a cold-air box should be equal to the combined areas of all the registers minus one sixth. There are 8 registers in a house, each 8 in. by 10 in. How large should the cold-air box be?

19. The sectional area of a cold-air box is 600 sq. in.; each register measures 8 in. by 10 in. How many registers are there in the house?

20. The cross section of a cold-air box measures 20 in. by 20 in. There are 6 equal registers in the house. What is the area of each?

21. It is said that the minimum dimensions of an ideal dining room are 11 ft. by $13\frac{1}{2}$ ft. How many square feet are contained in such a room?

22. An ideal dining room of maximum size measures 17 ft. by 22 ft. How many square feet are contained in such a room?

23. A kitchen, according to one authority, should measure 10 ft. by 12 ft. If the range measures 2 ft. by 3 ft. 14 in., the sink 1 ft. 6 in. by 3 ft. 15 in., two cupboards each 1 ft. 8 in. by 6 ft. 4 in., and the work table 4 ft. by 5 ft. 7 in., how many square feet are allowed for "walking" space?

24. It is said that stairs are well proportioned when 2 times the height of the riser, added to the tread, equals 24 in. The riser of such stairs is 7 in. What is the tread?

25. Measure the stairs in your home. How near do they come to the ideal measurements?

26. How large a piece of material must I have to make a bag $10'' \times 14''$ when finished, if I allow $2''$ for a heading and $\frac{1}{2}''$ on three sides for hems? Use the width of the material for the depth of the bag.

27. A waist pattern requires $3\frac{1}{2}$ yards of $27''$ material. How many yards of $36''$ material is required?

28. A sleeve pattern calls for $\frac{7}{8}$ yard of goods $36''$ wide. How much $24''$ material will it require?

29. *I buy a dress pattern which calls for* $5\frac{7}{8}$ yards of $44''$ goods. *If I wish to use* material 27 inches wide, how much must I b

CHAPTER VI

Polynomials. Multiplication of Polynomials

103. Polynomial by Monomial. Review multiplication of monomials, §§ 31–34. Give sign rule for multiplication. Give exponent rule for multiplication. Give sign rule for addition. Illustrate each.

104. In § 34, we found the product of a monomial by a monomial. We shall now extend multiplication to cover any number of terms.

A polynomial is simply a sum of monomial terms.

Hence, to multiply a polynomial by a number is to multiply each of its terms by that number, and find the sum of these products.

Ex. Multiply $5a^2+3ax-x^2$ by $2a$.

$$5a^2\cdot 2a=10a^3,\quad 3ax\cdot 2a=6a^2x,\quad -x^2\cdot 2a=-2ax^2.$$

Then, $\quad (5a^2+3ax-x^2)\cdot(2a)=10a^3+6a^2x-2ax^2.$

The work should be written in the following form:

$$\begin{array}{l} 5a^2+3ax-x^2 \\ \underline{2a} \\ 10a^3+6a^2x-2ax^2 \end{array}$$

Begin multiplication at the left.

EXERCISE 26

Find the following products. (Perform the numerical multiplication mentally, writing results only.)

1. $\begin{array}{l} 7a^2 + 3ab + 2b^2 \\ \underline{5a \qquad\qquad\qquad} \end{array}$

2. $\begin{array}{l} 5m^2 - 3cm + y \\ \underline{2c^3y \qquad\qquad\quad} \end{array}$

3. $\begin{array}{l} -4x^2 + 2xz - 5z^2 \\ \underline{-6x \qquad\qquad\qquad} \end{array}$

4. $\begin{array}{l} -14ad^2 + 15a^2d + 17d^3 \\ \underline{-13a \qquad\qquad\qquad\qquad} \end{array}$

5. $\begin{array}{l} 18x^2 - 17xy - 24y^2 \\ \underline{12x^4 \qquad\qquad\qquad} \end{array}$

6. In example 5, substitute $x = 1$, $y = 2$ in your multiplicand, multiplier, and product. Is the result what you might expect?

Any example in multiplication may be checked by substituting some numerical value for each letter as suggested in example 6.

7. Check each of the first five examples.

Multiply:

8. $32x^3z - 14xz^3$ by $16xz$.

9. $115xy^2 - 112x^3y$ by -12.

10. $1024a^2 + 612ab - 306b^2$ by $4ab^2$.

11. $(x+y)^2 + 6(x+y) + 9$ by $(x+y)$.

12. $(x+y)^3 + 3(x+y) - 4$ by $16(x+y)^2$.

13. $-24(x-y)^3 + 14(a+b)^2 - 21$ by 16.

14. $3(x+y)^2 + 12(x+y) + 18$ by $4(x+y)$.

15. Check example 14.

105. Polynomial by Polynomial. To multiply a polynomial by a polynomial is to multiply the multiplicand by each term of the multiplier, and add the partial products.

Ex. 1. Find the product of $2a + 3b$ by $3a - 5b$.

$$(2a + 3b)3a = 6a^2 + 9ab, \qquad (\S\ 103)$$
$$(2a + 3b)(-5b) = -10ab - 15b^2.$$

Adding these partial products

$$\begin{array}{l} 6a^2 + 9ab \\ \quad\;\; - 10ab - 15b^2 \\ \hline 6a^2 - \quad ab - 15b^2 \end{array}$$

we have

The work should be written as follows:

$$\begin{array}{rrr} 2a + 3b & & \\ 3a - 5b & & \\ \hline 6a^2 + & 9ab & \\ & -10ab & - 15b^2 \\ \hline 6a^2 - & ab & - 15b^2 \end{array}$$

The procedure is the same for any number of terms. The pupil will find the work more simple if both multiplicand and multiplier are arranged according to either the descending or ascending powers of some letter.

Ex. 2. Multiply $5x - 6x^2 + x^3 - 4$ by $-3x + x^2 - 1$.

Rearranging according to the descending powers of x,

$$\begin{array}{rrrrrr} x^3 & -6x^2 & +5x & -4 & & \\ x^2 & -3x & -1 & & & \\ \hline x^5 & -6x^4 & +5x^3 & -4x^2 & & \\ & -3x^4 & +18x^3 & -15x^2 & +12x & \\ & & -x^3 & +6x^2 & -5x & +4 \\ \hline x^5 & -9x^4 & +22x^3 & -13x^2 & +7x & +4 \end{array}$$

EXERCISE 27

Find the following products. Check each result:

1. $(5a + 3b)(2a + 4b)$.
2. $(6x + 2y)(5x - 3y)$.
3. $(7c + 2d)(7c - 3d)$.
4. $(7c + 2d)(7c - 2d)$.
5. $(x^3 + x^2 + 1)(x - 1)$.
6. $(5x^3 - 4x^2 + 3x - 2)(2x - 7)$.
7. $(x^2 + x + 2)(x^2 - x + 2)$.
8. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
9. $(a^2 + 2ab + b^2)(a + b)$.
10. $(a^2 - 2ab + b^2)(a - b)$.

Are the products in 9 and 10 alike?

11. Multiply $a^2 + 2a + 4$ by $a - 2$.
12. *Multiply* $5x^2 - 30x + 45$ by $5x - 15$.

13. $(a+b)(a+b)$.

14. $(a+2b)(a+2b)$.

15. $(x-2y)^2$.

16. $(x-y)(x-y)$.

Note the form of these results in examples 13–16. They will be useful later.

17. $(a+b)(a-b)$.

18. $(a+2b)(a-2b)$.

19. $(4x+y)(4x-y)$.

20. $(4x+3y)(4x-3y)$.

21. $(16c+15d)(16c-15d)$.

Note the form of the results in examples 17–21.

Write the following products by inspection:

22. $(c+2d)(c-2d)$.

23. $(5x+y)(5x-y)$.

24. $(5x+y)(5x+y)$.

25. $(3x+2y)(3x+2y)$.

26. $(2a+3c)(2a+3c)$.

27. $(5m+4k)(5m+4k)$.

Solve examples 28–32 mentally:

28. The side of a square is $a+b$. Find the area.

29. The side of a square is $2a-b$. Find the area.

30. The side of a square is $4x+3y$. Find the area.

31. The side of a square is $6c-4d$. Find the area.

32. The side of a square is $5x-2a$. Find the area.

33. In examples 28–32 find the dimensions and the area if $a=2$, $b=1$, $c=-1$, $d=3$, $x=4$, $y=-1$.

Find the areas of the following △ where $b=$ base and $a=$ altitude:

	b	a
34.	$6x+3y$,	$5x+7y$.
35.	$a+b$,	$a-b$.
36.	$x+8$,	$x-2$.
37.	$x+9y$,	$x+2y$.
38.	$x+15a$,	$x-3a$.

39. Compute the areas in examples 34–38 when $a=3$, $b=1$, $x=7$, $y=-6$.

40. Solve $16x^4+(8x^2-3x+14)(5x-2x^2+24)-336$.

41. Solve $[(x^2+3x-4)+(x^2-3x+4)](9x^2-6x+1)$.

42. Solve $(2x+3)(2x+3)^2$.

43. The edge of a cube is $4a+b$; find its volume.

44. The edge of a cube is $2x-5y$; find its volume.

45. The edge of a cube is $5x-14b$; find its volume.

46. Find the volumes in examples 43–45 if the letters have the same values as in example 33.

Find the areas of the following trapezoids: (B represents the lower base, b the upper base, and a the altitude).

	B	b	a
47.	$x+y$,	$x-y$,	2.
48.	$3x+2y$,	$2x+3y$,	$4x+4y$.
49.	$5x+2y$,	$3x-y$,	$6x+y$.
50.	$3x+5y$,	$7x-3y$,	$5y+x$.
51.	$7c+3d$,	$2c+d$,	$c-d$.
52.	$4c-8d$,	$c+5d$,	$c-3d$.
53.	$x+9y$,	$x-6y$,	$x-y$.
54.	$x+18$,	$x-15$,	$4y-x$.
55.	$c+8$,	$c-5$,	$c-2$.

56. Find the upper base, lower base, altitude, and area in examples 49–55, if $x=3$, $y=2$, $c=4$, $d=-5$. How do you account for your negative areas?

57. If 10′ be subtracted from the length of a rectangle, and the same amount is added to the breadth, the area will be *increased* by 100 square feet, but if 10′ be subtracted from the *breadth and 10′ added* to the length, the area will be diminished

by 300 square feet. Make a diagram of each rectangle, using a scale of $\frac{1}{16}'' = 1$ foot.

Division of Polynomials

106. Polynomial by Monomial. Review division of monomials, §§ 32–35. Give sign rule for division. Give exponent rule for division. Give rule for subtraction of monomials.

107. Since a polynomial is a sum of monomial terms, to divide a polynomial by a monomial, divide each term of the polynomial by the monomial and add the quotients thus found.

Ex. 1. Divide $81\,a^4 - 27\,a^2 + 18\,a$ by $9\,a$.

$$81\,a^4 \div 9\,a = 9\,a^3, \quad -27\,a^2 \div 9\,a = -3\,a, \quad 18\,a \div 9\,a = 2.$$

Then $(81\,a^4 - 27\,a^2 + 18\,a) \div 9\,a = 9\,a^3 - 3\,a + 2$.

The work should be written as follows:

$$\begin{array}{r|l} 9\,a) & 81\,a^4 - 27\,a^2 + 18\,a \\ \hline & 9\,a^3 - \;\;3\,a \;+\; 2 \end{array}$$

Ex. 2. Divide $21\,x^4 - 18\,x^3 + 5\,x^2 - 9\,x$ by $-3\,x$.

$$\begin{array}{r|l} -3\,x) & 21\,x^4 - 18\,x^3 + 5\,x^2 - 9\,x \\ \hline & -7\,x^3 + \;6\,x^2 - \frac{5}{3}\,x \;+ 3 \end{array}$$

Ex. 3. Divide $4\,a^2(2\,m+3) - 9\,a(2\,m+3) + 2\,m + 3$ by $2\,m+3$.

$$\begin{array}{r|l} 2\,m+3) & 4\,a^2(2\,m+3) - 9\,a(2\,m+3) + 2\,m + 3 \\ \hline & 4\,a^2 \qquad\qquad -9\,a \qquad\qquad +1 \end{array}$$

Note that in dividing one number by another, we are *simply removing from the dividend the factors found in the divisor.*

Ex. 4. $45 \div 3$.

$$45 = 3^2 \cdot 5.$$

$$\begin{array}{r|l} 3) & 3^2 \cdot 5 \\ \hline & 3 \cdot 5 \end{array}$$

After removing the 3 of the divisor $3 \cdot 5$ are left for the quotient.

Ex. 5. $45\,x^3y \div 9\,x^2$.

$$\begin{aligned} 45\,x^3y &= 3^2 \cdot 5 \cdot x \cdot x \cdot x \cdot y. \\ 9\,x^2 &= 3^2 \cdot x \cdot x. \end{aligned}$$

$$3^2 \cdot x \cdot x\overline{)3^2 \cdot 5 \cdot x \cdot x \cdot x \cdot y}$$
$$5 \cdot x \cdot y = 5\,xy$$

If one carries this principle in mind *no rule* for division by a monomial is necessary.

EXERCISE 28

1. $72\,x^7y \div 24\,x^5y$.
2. $4(a+b)^5 \div 4(a+b)^3$.
3. $(128\,x^6 - 80\,x^4 + 32) \div 16$.
4. $128\,x^6 - (80\,x^4 + 32) \div 16$.
5. $128\,x^6 - 80\,x^4 + 32 \div 16$.
6. $(128\,x^6 - 80\,x^4) + 32 \div 16$.
7. $[7\,x^4(2\,a+b)^3 - 12\,x^2(2\,a+b)^2 + 15\,x(2\,a+b)] \div -(2a+b)$.
8. Divide $14\,a(x-y) + 7\,a^2(x-y)^2 - 49\,a^3(x-y)^3$ by $-7\,a(x-y)$.
9. Divide $(a+b)a^2 + (a+b)2\,ab + b^2(a+b)$ by $a+b$.
10. Divide $144\,x^4y^2z - 729\,x^2y^3z^4 + 162\,xy^2z^5$ by $-3\,x^2y^2z$.

108. Polynomial by polynomial.

The product of x^2+x+2 by $2\,x+3$ is found by

$$\begin{aligned} &x^2(2\,x+3) + x(2\,x+3) + 2(2\,x+3) &(1)\\ &= 2\,x^3 + 3\,x^2 + 2\,x^2 + 3\,x + 4\,x + 6 &(2)\\ &= 2\,x^3 + 5\,x^2 + 7\,x + 6. &(3) \end{aligned}$$

Ex. 1. Required to divide $2\,x^3 + 5\,x^2 + 7\,x + 6$ by $2\,x+3$.

This means that $2\,x^3 + 5\,x^2 + 7\,x + 6$ is the product of two factors or sets of factors, one of which is $2\,x+3$. The problem is to find the other factor.

If $2\,x^3 + 5\,x^2 + 7\,x + 6$ is written in form (1), division can be performed as in example 3, § 107, but if the multiplication has been completed and *the partial products* added as in form (3), the factor required is not so *readily seen.*

It is evident from (1) that (3) is made up of partial products. If $2x+3$ is the divisor, x^2+x+2 is the quotient.

An examination of (1) shows that the first term of each partial product is the product of the first term of the divisor by the corresponding term of the quotient. Therefore we may form this rule:

1. *Arrange* both dividend and divisor according to the ascending or descending powers of the *same* letter.

2. Divide the *first term* of the dividend by the *first term* of the divisor.

3. Multiply *each term* of the divisor by the quotient found in 2.

4. Subtract the product found in 3 from the dividend, arranging the difference found in the same order as the dividend.

5. Divide the first term of the difference by the first term of the divisor. This gives a second term of the quotient.

6. Proceed in this manner, considering each difference as a new dividend until the first term of the difference is of lower degree than the first term of the divisor.

7. If there is a remainder, make it the numerator of a fraction whose denominator is the divisor, and annex with proper sign to the quotient.

$$
\begin{array}{llr|l}
 & & \text{DIVIDEND} & \text{DIVISOR}\\
 & & 2x^3+5x^2+7x+6 & \underline{2x+3}\\
\text{1st partial product,} & x^2(2x+3) = & \underline{2x^3+3x^2} & x^2+x+2\\
 & & 2x^2+7x+6 & \text{QUOTIENT}\\
\text{2d partial product,} & x(2x+3) = & \underline{2x^2+3x} & \\
 & & 4x+6 & \\
\text{3d partial product,} & 2(2x+3) = & \underline{4x+6} &
\end{array}
$$

Ex. 2. Divide x^3+3x-2 by $x-4$.

$$
\begin{array}{r|l}
x^3+3x-2 & x-4\\
\underline{x^3-4x^2} & \overline{x^2+4x+19+\dfrac{74}{x-4}}\\
4x^2+3x-2 & \\
\underline{4x^2-16x} & \\
19x-2 & \\
\underline{19x-76} & \\
\dfrac{74}{x-4} &
\end{array}
$$

EXERCISE 29

Verify each result:

1. Divide $x^2(2x+3)+8x(2x+3)+15(2x+3)$ by $2x+3$.

2. Divide $a^2(a+b)+2ab(a+b)+(a+b)^2$ by $a+b$.

3. The area of a rectangle is $x^2+8x+15$. The length is $x+5$. What is the breadth? What is the breadth if $x=2$?

4. Divide $y^2-2y-15$ by $y+3$.

5. Divide $y^2+2y-15$ by $y+3$.

6. Divide $y^2+2y-15$ by $y-3$.

7. Divide $y^2-8y+15$ by $y-3$.

8. $(5x^3-20x^2+15x-30)\div -5$.

9. $5x^3-(20x^2+15x-30)\div -5$.

10. $5x^3-20x^2+(15x-30)\div -5$.

11. $5x^3-20x^2+15x-30\div 5+3(2x-8)\div 2$.

12. $4(a+b)^2\div 2(a+b)+4(a+b)^2\div -2(a+b)$.

13. The radius of a circle is $x+3$. Find the area.

14. The area of a trapezoid is $2x^2+12x+18$; the sum of the bases is $4x+12$. Find the altitude. If the upper base is $x+7$, what is the lower base? If $x=3$, what are the dimensions? Can you draw the trapezoid if it is isosceles?

15. Divide x^3+3x^2+3x+1 by x^2+2x+1.

16. Divide x^3+3x^2+3x+1 by $x+1$. Divide the quotient by $x+1$. Compare results in examples 15 and 16.

17. Divide x^3+27 by $x+3$.

18. Divide a^3+1 by $a+1$.

19. Divide $8a^3-27b^3$ by $2a-3b$.

20. Divide x^5-32 by $x-2$.

21. Divide x^4-81 by $x-3$.

22. Divide x^4+81 by $x+3$.

23. Divide $x^2 - x - 72$ by $x - 9$.

24. The area of a trapezoid is $15x^2 - 34x + 16$; the altitude is $5x - 8$; one base is $2x + 3$. Find the other base.

25. The area of an isosceles triangle is $12x^2 + 32x - 35$; the base is $6x - 5$. Find the altitude, then construct the triangle when $x = 2$. Construct the triangle when $x = -2$.

26. The area of a rectangle is $4x^2 + 12x + 9$; the base is $2x + 3$. Find the altitude. Construct the rectangle. Compare the base and altitude of your drawing. What kind of a rectangle is it?

27. Divide the sum of $x^3 + 7x^2 + 35x + 19$ and $x^3 + 8x^2 - 13x - 34$ by the difference between $5x^2 + 2x - 7$ and $3x^2 - 3x - 4$.

28. Perform the following operation: $\dfrac{(3x+7)(x^2 - x - 12)}{x - 4}$.

29. $x^2(2x^3 + 9x^2 - 71x - 120) \div (x^2 - 3x - 40)$.

30. $x^2(2x^3 + 9x^2 - 71x - 120) \div x^2 + 3x - 40$.

31. Divide $a^2 + 2ab + b^2$ by $a + b$. Is $a^2 + 2ab + b^2$ a square or a rectangle? Define a square.

32. $(100x^4 - 229x^2 + 9) \div (5x - 1)$.

33. Divide $100x^4 - 229x^2 + 9$ by $4x^2 - 9$.

34. Divide $100x^4 - 229x^2 + 9$ by $25x^2 - 1$.

35. Divide $100x^4 - 229x^2 + 9$ by $5x + 1$.

36. Divide $100x^4 - 229x^2 + 9$ by $2x + 3$.

37. Divide $100x^4 - 229x^2 + 9$ by $2x - 3$.

38. What are the factors of $100x^4 - 229x^2 + 9$? of $4x^2 - 9$? of $25x^2 - 1$?

39. Divide $x^3 + 6x^2 - x - 30$ by $x - 2$.

40. Divide $x^3 + 6x^2 - x - 30$ by $x + 3$.

41. Divide $x^3 + 6x^2 - x - 30$ by $x + 5$.

42. What are the factors of x^3+6x^2-x-30? What does the product of these factors equal?

43. The length, breadth, and thickness of a rectangular solid are $x+5$, $x+3$, and $x-2$, respectively. Find its volume. What are its dimensions when $x=3$? $x=2$? $x=1$? $x=0$? $x=-1$?

44. Divide x^2-64 by $x-8$.

45. Divide x^2-64 by $x+8$.

46. What are the factors of x^2-64?

47. The volume of a circular cylinder is $(16x^3-84x^2+120x-25)\pi$. The radius of the base is $2x-5$. Find the altitude. Find the dimensions when $x=3$; 4; 5.

108. The various symbols of algebra enable us to express many operations by using these symbols in place of words. In other words, the symbols are the shorthand, the stenography, of mathematics.

For example: If we wish to indicate the subtraction of $2x-1$ from $4x+7$, we may write:

(*a*) from $4x+7$ take $2x-1$; or we may write:
(*b*) $(4x+7)-(2x-1)$;

the two expressions being identical, and read in the same way. In the next exercise be sure you translate the algebraic language into the English before attempting to solve the problems.

REVIEW

1. $(2x+3)(2x-7)+(3x-1)(2x+5)$.
2. $(2x+3)(3x+2)-(4x+15)$.
3. $(4x-1)(x+4)-(x-4)(4x+1)$.
4. $x^3-[3x^2+(3x-1)]-[x^3+3x^2-(3x-1)]$.
5. $4(x-5)(x-2)-4$.

6. $7 + 3 \div 3 - 4.$

7. $2 + (8 \div 4) \cdot 5 - 8.$

8. $6 - 90 \div (3 \cdot 10) + 2 - 8.$

9. $(2x - 7)x - (3x + 4)x.$

10. Two right triangles are equal if two legs of the one are equal respectively to two legs of the other. Prove.

11. In a rectangle the opposite sides are equal. Prove that the diagonal divides a rectangle into two equal triangles.

12. The bisectors of the base angles of an isosceles triangle are equal. Prove.

13. $[(a + b)^5 + 6b(a + b)^4 - 12(a + b)^3] \div (a + b)^2.$

14. $(4a^2 - b^2) - (2a + b)(2a - b).$

15. $(64c^2 - 25d^2) - (8c + 5d)(8c - 5d).$

16. $(5x + 2y)^2 - (5x - 2y)(5x - 2y).$

17. $(5x + 2y)(5x - 2y) - (25x^2 - 4y^2).$

18. $(7x - 3z)^2 - (7x + 3z)^2.$

19. $(x + 5)(x - 2) - (x - 5)(x + 2) - (x - 5)(x - 2)$
$- (x + 5)(x + 2).$

20. $(a + b)^3.$

21. $(c + 7)^3.$

22. $(a^2 + ab + b^2)(a^2 - ab + b^2).$

23. $(a^2 + ab + b^2)(a^2 + ab + b^2).$

24. Find the area of a trapezoid whose upper base, lower base, and altitude are $x + 7$, $2x + 8$, $3x + 2$ respectively. If $x = 1$ and the trapezoid is isosceles, construct the trapezoid.

25. Find the prime factors of $(25)^2$, $(24)^2$, $(72)^2$.

26. What are the prime factors of $(12)^6$? $(144)^5$?

27. What are the prime factors of 1728? of $(39)^3$?

28. *Factor* $(81)^4$, $(8)^5$, $(16)^{10}$.

29. Factor $27 \cdot 81 \cdot 729$.

30. Factor $72 \cdot 64 \cdot 36$.

31. Factor $45\, x^3y^2$. How many prime factors in this number?

32. Factor $81\,(a+b)^5$.

33. Factor $27\,(2\,x-y)^3$. If $x=3$ and $y=1$, what are the prime factors of this product? How do they differ from the prime factors of $(15)^3$?

Supplemental Applied Mathematics

1. One egg weighs 2 ounces. 57 % of the egg is white, 32 % yoke, 11 % shell. Find the weights of the whites, yolks, and shells of one dozen eggs.

2. One egg weighs 2 ounces. The shells of 2 eggs weigh one ounce. What per cent of the whole egg is the edible portion?

3. The edible portion of 2 eggs measures $\frac{3}{8}$ of a cup. It takes 9 whites of eggs to measure 1 cup. How many yolks of eggs does it take to make 1 cup?

4. Beaten whites of 4 eggs measure $2\frac{1}{8}$ cups. What is the per cent of increase in quantity of beaten over unbeaten whites?

5. Beaten yolks of 3 eggs measure $\frac{1}{8}$ cup. What is the per cent of increase in quantity of beaten over unbeaten yolks?

6. One egg, yolk and white beaten together, measures 4 tablespoonfuls. How much greater is the increase in quantity when yolks and whites of 4 eggs are beaten separately?

7. According to Hutchison, experiments as to the difference in time of digesting eggs cooked in various ways show that 2 soft "boiled" eggs leave the stomach in $1\frac{3}{4}$ hours, and 2 hard "boiled" eggs leave the stomach in 3 hours. If 2 eggs are eaten on each of 4 days a week, how many more hours' work a *month will* the stomach have in digesting hard than soft *"boiled" eggs?*

8. An egg was boiled for 3 minutes. After artificially digesting for 5 hours in a pepsin solution, it contained 8.3 % undigested protein. An egg was cooked in water at 180° F. for 5 minutes. It was entirely digested after 5 hours in a pepsin solution. The edible portion of eggs contains 13.4 % protein. Find the weight of undigested protein from 1 dozen eggs if they are soft "boiled" rather than soft "cooked."

9. Steel is 7.83 times as heavy as water. Find the weight of one cubic inch of steel.

10. A gas range has four burners, each of which burns .65 cubic feet per minute and two oven burners, each burning 1.032 cubic feet per minute. Find the cost per hour of running the stove when all burners are on full, gas at 65 ¢ per thousand.

11. A house is heated by a gas furnace containing four burners and two pilot lights. Each burner consumes one cubic foot in two minutes; each pilot one cubic foot in eight minutes. Find gas bill for February at 30 ¢ per thousand cubic feet. The pilots burned all the time. Two burners burned from 6 A.M. to 10 P.M. and four burners were running two hours in the morning and two hours in the evening each day.

12. According to one authority, a family of five living on $2000 to $4000 per year should spend 25 % of that sum for food; 20 % for rent; 15 % for operating expenses, such as fuel, wages, etc., 15 % for clothing; and 25 % for higher life, *i.e.* books, travel, charity, saving, and insurance. What amount should be used for each item if a family has an annual income of $3000?

13. A family living on $1000 to $2000 per year requires 25 % for food; 20 % for rent; 15 % for operating expenses; 20 % for clothing; and 20 % for higher life. If a family lives on $1500 per year, what amount should be spent for each item?

14. From a family income of $800 to $1000 per year, 30 % should be spent for food; 20 % for rent; 10 % for operating expenses; 10 % for clothing; and 20 % for higher life.

How much can be spent for each item when the income is \$900?

15. A family living on \$500 to \$800 per year should spend 45% for food; 15% for rent; 10% for operating expenses; 10% for clothing, and 10% for higher life. If a family's annual income is \$650, how much should be spent for each item?

16. From an annual income of less than \$500, 60% should be spent for food; 15% for rent; 5% for operating expenses; 10% for clothing; and 10% for higher life. If the income of a family is \$425 per year, how much can be spent for each item per year and month?

17. The grocery bill of a family living on \$3000 per year should be 25% of that sum; of a family living on \$700 a year, 45% of that sum. Find the difference in the monthly grocery bill of each?

18. 20% of a twelve-hundred-dollar income should be spent for clothing, and 15% of a nine-hundred-dollar income. Find the difference in the amounts spent for these items by two families having these incomes.

19. There are 49 lb. flour in a fourth barrel, or an ordinary sack of flour. 1 lb. flour measures 4 cups. If a family uses 5 loaves bread per week, and it takes $3\frac{1}{2}$ cups of flour to make one loaf, how many months will a sack of flour last?

20. Find the cost of 4 loaves of bread, requiring 1 hour for baking and containing the following: $3\frac{1}{2}$ qt. flour, 1 yeast cake, 2 tb. lard, 4 t. salt, 4 t. sugar. Flour costs \$2.00 per one-fourth barrel; yeast 2¢ per cake; lard 15¢ per pound (2 c. in a pound), the sugar and salt \$.0024; gas 70¢ per 1000 ft., the oven burner burning 39 cu. ft. per hour.

(Tb. = tablespoon, t. = teaspoon, c. = cup.)

21. I buy 28-inch material for handkerchiefs. How many yards would I have to buy to make 9 dozen handkerchiefs, cut 14 inches square?

22. What is the size of the finished handkerchief if a $\frac{1}{4}$-inch hem is made on four sides?

23. If I paid 2 cents a yard for stitching, how much would the work on 6 dozen handkerchiefs cost?

24. How much lace would it take to sew around 9 dozen handkerchiefs if you allowed 4 inches extra on each handkerchief for fullness?

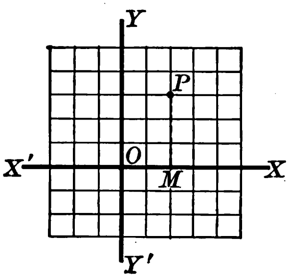
25. If the material cost 60 cents a yard, and lace 15 cents a yard, how much would the handkerchiefs in problem 24 cost, including 5 spools thread at 5 cents a spool and 2 cents a yard for stitching?

CHAPTER VII

Graphs, the Algebra of Lines. Parallels and their Uses

109. In §§ 43–49 we studied and solved simultaneous equations. We will now study these equations from a geometric standpoint.

In § 25 we determined that if toward the right were positive, toward the left should be negative. Suppose we measure our positive and negative values from two lines intersecting at right angles. Suppose further, that horizontal measurements shall be x-measurements and vertical measurements be y-measurements, upward being positive and downward being negative.



Ex. 1. Find the point where $x = 2$, $y = 3$.

Measuring $OM = 2$ and $MP = 3$, we have the point P satisfying the required condition.

EXERCISE 30

In the same manner locate the following points:

1. $x = 4, y = 2.$
2. $x = 3, y = 5.$
3. $x = 2, y = -3.$
4. $x = -1, y = 4.$
5. $x = -1, y = -4.$
6. $x = -1, y = 2.$
7. $x = 5, y = -1.$
8. $x = 0, y = 2.$

9. $x=2, y=0$. 10. $x=0, y=-3$.

11. $x=-2, y=0$.

12. Are these points on the same circumference?

$x=3, y=4$; $x=4, y=3$; $x=-3, y=4$;
$x=4, y=-3$; $x=-4, y=3$; $x=3, y=-4$;
$x=-3, y=-4$; $x=-4, y=-3$.

(Take your center at O.)

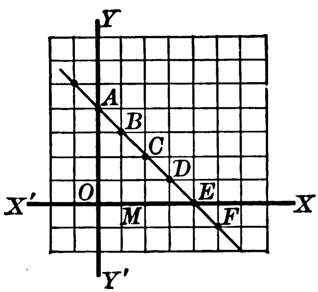
110. In the equation $x+y=4$, x and y are so related that their sum must always equal 4. x and y then may take any values whose sum is 4. To find such pairs of values, solve $x+y=4$ for either x or y. We will solve for y.

$$y=4-x.$$

Give x a set of values, say 0, 1, 2, 3, etc., and find the corresponding values of y. If $x=0$, $y=4$; if $x=1$, $y=3$, etc.

It is convenient to write these values in two columns headed x and y. See below.

x	y
0	4 (A)
1	3 (B)
2	2 (C)
3	1 (D)
4	0 (E)
5	−1 (F)
−1	5 etc.



Now plot these values on the same pair of axes, just as we did in exercise 27.

In this way we obtain the points A, B, C, D, etc.

If these points are connected by a smooth curve, any point on the curve will correspond to a point satisfying an x and a y of $x+y=4$.

The values of x and y which locate the position of a point are called **coördinates** of a point. The x-measurement, *e.g.*, OM, is called the **abscissa**, the y-measurement, *e.g.*, MB, is the **ordinate.** The point O is called the **origin.**

If each term of the equation is of the **first degree,** the curve $ABCDEF$ is a straight line and the equation is called **linear.**

This curve is called the **graph** or **geometric picture** of the equation.

111. The **degree of a term** is determined by the sum of the exponents of the letters in it. In an equation the term of highest degree determines the degree of the equation.

$x^3y + xy^2 - x^3$ is an expression of the fourth degree in xy and of the third degree in x.

$x^3 - 3x^2 = 8$ is an equation of the third degree.

$2x + a^2 = b^3 - c$ is an equation of the first degree in x.

EXERCISE 31

Find the graphs of the following equations:

1. $x + y = 6.$	5. $2x - y = 5.$
2. $2x + y = 8.$	6. $3x - 2y = 5.$
3. $x - y = 4.$	7. $x + 4y = -3.$
4. $4x - 3y = 12.$	8. $x + 2y = -4.$

112. If a pair of simultaneous equations (§ 43) are plotted on the same axes, their graphs will usually intersect. In this case the coördinates of the point of intersection are the same as the values that are found for x and y when the equations are solved.

This explains in a geometric way the name **simultaneous** equations. The x and y must at the *same time* have values *which satisfy both* equations.

Ex. Solve by graphs:

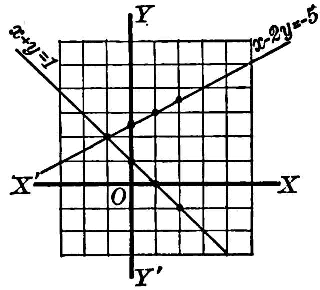
$$x - 2y = -5, \quad (1)$$
$$x + y = 1. \quad (2)$$

From (1)

x	y
0	$+\frac{5}{2}$
1	$+3$
2	$\frac{7}{2}$
-1	2

From (2)

x	y
0	1
1	0
2	-1
-1	2



In the above problem the lines (1) and (2) intersect at $x = -1$, $y = 2$. If the equations are solved by the principles of §§ 46–48, the same results are obtained.

113. Note that in these equations x and y *vary.* Any change in one of these numbers causes a change in the other. For this reason x and y are the *variables* in the equation. (Compare § 44.) Letters and numbers whose values do not change in an equation are called **constants.** The number which is *independent* of the variables is called the **absolute** term.

Sometimes the graphs of equations will not intersect. The equations are then said to be **inconsistent.**

Ex. Plot the equations $x + y = 4,$
$$\underline{x + y = 2.}$$

The graphs of these equations do not intersect. This is because the lines are parallel.

114. Parallel Lines. Two straight lines are said to be parallel when they have the same direction. It is evident that if they are drawn through two different points they are everywhere equally distant and can never meet.

115. Some of the graphs of equations do not intersect because they coincide.

Ex. Plot $2x + 3y = 3$,
$4x + 6y = 6$.

These graphs coincide. Equations of this kind are said to be **equivalent**. See example 2, exercise 18.

Equivalent equations and *inconsistent* equations have no solution.

EXERCISE 32

Solve by means of graphs. In each case solve algebraically, also compare the result with the graphical solution. Note that if the roots are fractional, the graphical solution is only approximately correct.

1. $x + y = 7$,
   $2x - 3y = -6$.

2. $3x + 4y = 11$,
   $4x - 3y = -2$.

3. $x + 2y = 13$,
   $2x - y = 1$.

4. $x + 2y = 5$,
   $2x + y = 4$.

5. $\frac{x}{2} - \frac{y}{4} = 2$,
   $\frac{3x}{4} - \frac{y}{3} = \frac{19}{6}$.

6. $5x + 8y = 13$,
   $6x - 5y = 1$.

7. $4x - 7y = 18$,
   $6x + 5y = -4$.

8. $9x - 4y = 1$,
   $6x + 2y = 3$.

9. $5x - 4y = 2$,
   $4x + 5y = -2$.

What angle is formed by the lines in examples 2, 3, 9?

10. $2x - 5y = 3$,
    $3x - 7\frac{1}{2}y = 4\frac{1}{2}$.

11. $8x + 2y = 1$,
    $4x + y = \frac{1}{2}$.

12. $2x + y = 4$,
    $2x + y = 6$.

13. $5x - 3y = 4$,
    $5x - 3y = 5$.

14. $4u - 7v = -3$,
    $u + 3v = 4$.

What relations exist between the three lines:

15. $4x + 3y = 2$,
    $3x - 4y = 5$,
    $3x - 4y = 8$.

16. $5t + u = 11,$
$\underline{3t - 2u = 4.}$

17. $2m + 3k = 19,$
$\underline{3m - 4k = 3.}$

18. $6v - 8w = 5,$
$\underline{4v + 5w = 7.}$

Parallels and their Uses

116. Parallel lines (§ 114) are of great importance in geometry.

117. We assume that but one line can be drawn through a given point parallel to a given line.

118. Prove that *in the same plane two perpendiculars to the same line are parallel.*

Hint. Draw the line $x + y = -4$. At points $x = -4$, $y = 0$, and $x = 0$, $y = -4$, erect perpendiculars to this line. To prove that these perpendiculars are parallel, suppose that they meet if produced, then read Theorem IX. The method of proof you use here is called **Indirect.** (§ 102.)

Compare your *given* equation, $x - y = -4$, and $x - y = 4$, with the three lines of your figure. Are similar relations found in exercise 32?

Problem

119. *To draw a straight line parallel to a given straight line.*

1. Draw an indefinite line AB.

2. Choose a point P without line AB through which to draw the parallel.

3. Draw AP and extend it to some point K.

4. At P construct an angle equal to $\angle A$.

5. PZ, one side of $\angle ZPK$ is the required parallel.

THEOREM XIV

120. *Two lines parallel to the same line are parallel to each other.*

Draw line a. Draw lines b and c parallel to a. We then have

Given two lines b and c parallel to line a.

To prove line $b \parallel$ line c.

Proof. 1. If b and c are not parallel they will, if produced, meet at some point P.

2. The rest of the proof is left to the pupil. (See § 117.)

THEOREM XV

121. *A line perpendicular to one of two parallels is perpendicular to the other.*

Draw line $a \parallel$ line b. Draw line $c \perp$ line b, intersecting line a at K. We now have

Given lines a and b parallel, and line $c \perp$ line b.

To prove line $c \perp$ line a.

Proof. 1. Draw line m through K perpendicular to c.

2. Then, lines b and m are $\perp$ to c, and b and m are $\parallel$. (§ 118.)

3. Then, m and a coincide. (§ 117.)

4. Hence, $c \perp a$.

122. The angle opposite the base of a triangle is the *vertical angle.* (Any side may be the base.)

The vertex of the vertical angle is the **vertex of the triangle.**

The **altitude** of a triangle is the perpendicular from the vertex to the base.

An **exterior angle** of a triangle is the angle formed by any side of a triangle and the adjacent side produced.

EXERCISE 33

1. In an isosceles triangle draw the exterior angle at the base: at the vertex.

2. Draw three altitudes of an isosceles triangle.

3. Draw three altitudes of an equilateral triangle.

4. Draw three altitudes of an obtuse angled triangle.

5. Draw three altitudes of a right triangle.

6. Draw the three bisectors of the angles of the triangles in examples 2 to 5.

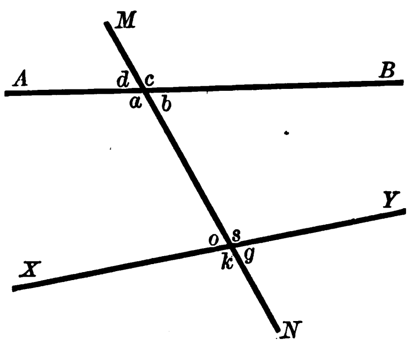
7. Do the three altitudes of these triangles ever meet in a point? Do they always meet in a point?

8. Do the altitudes and the bisectors ever coincide? If so, when?

9. Do the three bisectors ever meet in a point? Do they always meet in a point? Later in the course you will be called upon to prove your above conclusion.

123. If two lines AB and XY are cut by a third line MN, MN is said to be a **transversal.**

This transversal makes with the other two lines eight angles which have special names, names which refer to the position of the angles with respect to the lines. For example, a, b, o, s, are between lines AB and XY, and are called **interior angles.** The remaining four are **exterior angles.**



a and s being on *opposite* sides of the transversal are **alternate-interior** angles. Likewise o and b are alternate-interior angles.

c and s, on the *same* side of the transversal are **exterior-interior** angles. Locate the other exterior-interior angles.

d and g are **alternate-exterior** angles. Find the other alternate-exterior angles.

THEOREM XVI

124. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*

Draw $AB \parallel XY$, and MN intersecting AB and XY at O and K, respectively. Let $\angle KOA = c$, and $\angle YKO = d$. We then have

Given $\parallel$s AB and XY cut by MN forming alternate-interior angles c and d.

To prove $\angle c = \angle d$.

Proof. 1. Through Z, the middle point of OK, draw a $\perp$ to AB, meeting AB at F, and XY at E. Let $\angle FZO = g$ and $\angle KZE = i$.

2. $EF \perp XY$. (§ 121.)

3. In rt. $\triangle$s OFZ and KEZ,

$$g = i. \quad (\text{§ } 91.)$$
$$KZ = ZO. \quad (\text{Constr.})$$

4. Hence, $\triangle OFZ = \triangle KEZ$. (?)

5. Then, $\angle c = \angle d$. (§ 79.)

This is the fundamental proposition in parallel lines.

EXERCISE 34

1. If two parallels are cut by a transversal, the exterior-interior angles are equal. (Use the equations derived from the Theorems XVI and VII.) NOTE: Remember in proving any geometric statement you must give authority (the why) for each step you take.

2. If two parallels are cut by a transversal, the sum of the interior angles on the same side of the transversal is equal to two right angles.

3. If two parallels are cut by a transversal, the alternate-*exterior angles are* equal.

4. If two parallels are cut by a transversal, the sum of the exterior angles on the same side of the transversal is equal to two right angles.

THEOREM XVII

125. *If two lines are cut by a transversal, and the alternate-interior angles are equal, the lines are parallel.*

Draw a line AB. Draw a line MN intersecting AB at O. Through any point K on MN, draw line XY, making an angle YKO equal to KOA. Use same notation for angles as that in Theorem XVI. We have

Given two lines AB and XY cut by MN making $\angle c = \angle d$.

To prove $AB \parallel MN$.

Proof. 1. Through K draw $GH \parallel AB$.

2. Then, $\angle OKH = \angle c$. (Theorem XVI.)
3. But $\angle d = \angle c$. (By Hyp.)
4. Then, $\angle OKH = \angle d$. (Ax. 8.)
5. Hence, lines GH and XY coincide.
6. Therefore, $XY \parallel AB$.

This theorem is the *converse* of Theorem XVI. That is, the hypothesis and conclusion of the two theorems are interchanged. Theorems XVI and XVII are very important.

EXERCISE 35

1. If two lines are cut by a transversal and the exterior-interior angles are equal, the lines are parallel.

2. If two lines are cut by a transversal, and the sum of the interior angles on the same side of the transversal is equal to two right angles, the lines are parallel.

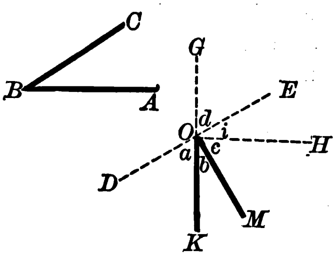
3. If two lines are cut by a transversal, and the alternate-exterior angles are equal, the lines are parallel.

4. If two lines are cut by a transversal, and the sum of the exterior angles on the same side of the transversal is 180°, the lines are parallel.

5. If two angles have their sides parallel, each to each, they are either equal or supplementary. (*Hint.* Produce one side of each angle, if necessary, until the lines intersect.)

6. If two angles have their sides perpendicular, each to each, they are either equal or supplementary.

Draw $\angle ABC$, also $\angle KOM$ whose sides are perpendicular, each to each, to the sides of $\angle ABC$. We have



Given $\angle s\ ABC$ and KOM with side $AB \perp KO$, and $BC \perp OM$.

To prove $\angle B$ equal to or supplementary to $\angle KOM$.

Proof. 1. *Through O draw $DE \perp OM$, D and E being on opposite sides of O; produce KO to G, and draw $OH \perp GK$. Call $\angle GOE$, d; $\angle HOE$, i; $\angle MOH$, c; $\angle KOM$, b; $\angle DOK$, a.

2.	Then $DE \parallel BC$.	(?)
3.	And $OH \parallel BA$.	(?)
4.	Therefore $\angle i = \angle B$.	(?)
5.	$i + c = 90°$.	(?)
6.	$b + c = 90°$.	(?)
7.	Then $i = b$.	(?)
8.	And $\angle B = \angle b$.	(?)

9. If OK were drawn in the opposite direction, namely OG, $\angle MOG$ would be the supplement of $\angle B$.

7. Two triangles have their sides mutually perpendicular. Show that they are mutually equiangular.

8. Two triangles have their sides mutually parallel. Show that they are mutually equiangular. Are the triangles equal?

* *Remember that all* **additional lines drawn in a figure must be dotted lines.**

Theorem XVIII

126. *The sum of the angles of a triangle is equal to two right angles.*

Given $\triangle ABC$.

To prove that $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

Proof. 1. Produce side BC to K. Draw $CM \parallel BA$, and on same side of BC as BA.

Let $\angle ACB = x$, $\angle ACM = y$, $\angle MCK = z$.

2. $x + y + z = 2$ rt. $\angle$s. (§ 76, *c*.)
3. $\angle B = z$ (?)
4. $\angle A = y$ (?)
5. $x = x$ (?)
6. Add equations 3, 4, and 5.
7. Compare equations 2 and 6.
8. Hence?

EXERCISE 36

1. Prove Theorem XVIII by drawing a line through the vertex, parallel to the base. (Do not draw any other lines.)

2. Prove two right triangles equal if a leg and an acute angle of the one are equal, respectively, to the leg and acute angle of the other.

3. Prove that the exterior angle (§ 122) of a triangle is equal to the sum of the two opposite interior angles. (You will need this theorem very often.)

4. Prove that the exterior angle of a triangle is greater than either of the opposite interior angles.

5. How many right angles can a triangle have?

6. How many obtuse angles can a triangle have?

7. One angle at the base of an isosceles triangle is $36°$. Find the other angles.

8. The vertical angle of an isosceles triangle is 120°. The base angles are bisected. Find the angles formed by the bisectors.

9. The angle formed by the bisectors of the base angles of an isosceles triangle is 100°. Find the angles of the triangle.

10. One angle of a right triangle is 45°. Compare the legs of the triangle.

11. (*a*) Construct an equilateral triangle.
(*b*) Construct an angle of 30°.

12. Construct an angle of 45°.

13. An exterior angle at the base of an isosceles triangle formed by producing the base is 108°. Find the angles of the triangle.

14. Could the exterior angle in example 13 be 89°? 90°? 91°? Why?

15. An exterior angle formed at the vertex of an isosceles triangle by producing one of the legs is 129°. Find the angles of the triangle.

16. Find the exterior angle at the base of an equilateral triangle.

17. Prove 6 of Exercise 35 by producing MO to meet BC, and producing BA to meet OK produced.

127. Develop the following synopsis:

Two triangles are equal if (*a*)
(*b*)
(*c*)

Two right triangles are equal if (*a*)
(*b*)
(*c*)
(*d*)

Keep this synopsis always in your mind.

Theorem XIX

128. *If two angles of a triangle are equal, the triangle is isosceles.*

Draw a line CD. At C and D construct equal angles. Let the sides of these angles meet at P. Call PC, d, and PD, c. We then have

Given $\triangle\ CDP$ with $\angle C = \angle D$.

To prove $c = d$.

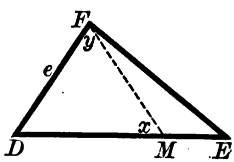
Proof. 1. Draw a perpendicular from P to the base.

2. Prove the $\triangle$s formed are equal.

Theorem XX

129. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*

Draw $\triangle\ DEF$ making $DE > DF$. Call DE, f, and DF, e. We now have



Given $\triangle\ DEF$ with $f > e$.

To prove $\angle EFD > \angle E$.

Proof. 1. On DE take $DM = e$, and draw FM.

Call $\angle DMF$, x, and $\angle MFD$, y.

(Note in this construction that D is the angle not involved in the statement, and from D we measure the distance DM.)

2. $x = y$. (Theorem III.)
3. $\angle EFD > y$. (Ax. 7.)
4. $\angle x > \angle E$. (Ex. 36, 4.)
5. $\therefore y > \angle E$.
6. $\therefore \angle EFD > \angle E$. (?)

EXERCISE 37

1. The perpendicular is the shortest line from a point to a line. (Draw the perpendicular and any other line from the given point to the given line.)

2. Two isosceles triangles are equal if the base and one base angle of one are equal, respectively, to the base and one base angle of the other.

3. The hypotenuse in a triangle is greater than either leg.

4. Find the sum of the angles of a quadrilateral. (Theorem XVIII.)

5. If the lines are drawn from a point within a triangle to the extremities of one side, the angle included by them is greater than the angle included by the other two sides. (Use the figure of Theorem VI, and apply Exercise 36, Example 4.)

Theorem XXI

130. *Any point in the bisector of an angle is equidistant from the sides of the angle.*

Draw $\angle DEF$. Bisect $\angle E$.

From P, or any point in the bisector, draw PM and $PK \perp$ ED and EF, respectively, meeting ED at M and EF at K.

Call PM, d_1 and PK, d_2. We now have

Given $\angle DEF$, and PE bisecting DEF, also, d_1 and d_2 $\perp$s from any point, P, in the bisector, to sides ED and EF, respectively.

To prove $d_1 = d_2$.

Proof. Show that $\triangle\ EPK = \triangle\ EMP$.

131. This bisector is sometimes called the **locus of points** equidistant from the sides of the angle. A locus may be defined as *a point or line* which fulfills conditions imposed upon it, no

other point or line meeting these conditions. Thus, in § 130 no point not in the bisector will satisfy the conditions of the theorem. The center of a circle is the locus of all points in a plane equidistant from the circumference.

EXERCISE 38

1. Show that every point within an angle, and equally distant from the sides of the angle, lies in the bisector of the angle.

2. The bisectors of the base angles of an isosceles triangle form with the base an isosceles triangle.

3. The point of intersection of the bisectors of the base angles of a triangle lies in the bisector of the vertical angle.

4. Find the point in the base of a triangle which is equidistant from the other two sides of the triangle. Is this point ever the middle of the base?

5. Show that if lines are drawn from the middle point of the base of an isosceles triangle respectively perpendicular to the legs of the triangle, two equal triangles are formed.

6. If from any point in the base of an isosceles triangle, parallels to the legs are drawn, two isosceles triangles are formed. Are the triangles ever equal?

7. Show that in a right triangle, if one angle is 30°, the hypotenuse is twice the shorter side. (Bisect the vertical angle of an equilateral triangle.)

8. Find a point which is equidistant from two intersecting lines.

9. If three lines intersect, but do not pass through the same point, find a point, if any such exists, which is equidistant from all three lines.

10. Bisect the exterior angles at the base of a triangle, and show that these bisectors meet in a point of the bisector of the angle at the vertex of the triangle.

11. Bisect the exterior angle at the vertex of an isosceles triangle. Show that this bisector is parallel to the base of the triangle.

12. Through the vertex of an isosceles triangle draw a parallel to the base. Show that this line bisects the exterior angle formed by extending one of the equal sides through the vertex.

13. Through the middle point of one leg of an isosceles triangle draw a parallel to the other leg. Through the vertex draw a parallel to the base. Show that two equal triangles are formed. How do you find the middle point of one leg?

14. Find a point equidistant from two parallel lines. Is there more than one such point?

15. If two lines intersect, the bisectors of two adjacent angles formed are mutually perpendicular.

THEOREM XXII

132. *If two triangles have two sides of one equal respectively to two sides of the other, and the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*

Draw $\triangle ABC$ and KLM, having sides $AC(b)$ and $CB(a)$ respectively equal to $MK(l)$ and $ML(k)$, and $\angle C > \angle M$. We now have

Given $\triangle ABC$ and KLM, with

$$b = l,\ a = k.$$

$$\angle C > \angle M.$$

To Prove $AB(c) > KL(m)$.

Proof. 1. Apply $\triangle KLM$ to $\triangle ABC$ so that l will coincide with b, L falling at L'. Draw CO, bisecting $\angle L'CB$, meeting AB at O. Draw AL'

2. $$AO + OL' > AL'. \qquad (?)$$

3. In $\triangle\!\!\triangle$ $CL'O$ and COB,

$$CL' = CB. \qquad (?)$$

$$CO = CO.$$

$$\angle L'CO = \angle OCB. \qquad (?)$$

4. Hence, $\triangle CL'O = \triangle COB$, (?)

and $OL' = OB$. (?)

5. Then, $AO + OL' = AO + OB$. (?)

6. Hence, $AO + OB > AL'$, (?)

or $c > m$.

The converse of this theorem is also true. State the converse.

Four-sided Figures

133. A **quadrilateral** is a portion of a plane bounded by four lines.

If the opposite sides are *parallel*, the figure is a **parallelogram**. Draw the figure.

If two sides are parallel and the other two sides not parallel, the figure is a **trapezoid**. Draw a trapezoid. In an **isosceles trapezoid** the non-parallel sides are equal.

If no two sides are parallel, the figure is a **trapezium**.

A **rhomboid** is a parallelogram whose adjacent sides are not equal and whose angles are oblique. Illustrate.

A **rhombus** is an equilateral parallelogram whose angles are oblique.

A **rectangle** is a parallelogram whose angles are right angles.

A **square** is an equilateral rectangle.

The **diagonal** of a quadrilateral is a line drawn from one vertex to the opposite vertex.

Begin lettering a four-sided figure at the lower left-hand corner and read counter-clockwise.

Make a synopsis classifying quadrilaterals under three general heads.

Theorem XXIII

134. *In a parallelogram the opposite sides are equal, and the opposite angles are equal.*

Draw a parallelogram *ABCD*. Draw *BD*. Prove the two triangles formed are equal. Then use § 79.

This is the fundamental theorem in parallelograms.

EXERCISE 39

1. The diagonal of a parallelogram divides it into two equal triangles.

2. The diagonals of a parallelogram bisect each other.

3. Parallel lines included between parallel lines are equal.

4. From two opposite vertices of a parallelogram draw perpendiculars to the diagonal drawn between the other vertices. Show that two pairs of equal triangles are formed.

5. Show that the lines connecting the middle points of the opposite sides of a parallelogram bisect each other.

6. The diagonals of a rhombus bisect each other at right angles.

7. In a certain parallelogram the diagonals bisect its angles. One side of the parallelogram is 8′. Find the other sides.

8. In an isosceles trapezoid draw perpendiculars from the extremities of the shorter base to the longer base. Show that two equal triangles are formed.

9. The diagonals of a rectangle are equal.

10. In an isosceles triangle *ABC*, the equal angles *A* and *B* are bisected. These bisectors form an angle of 120°. The leg of the triangle *ABC* is how many times its base?

Theorem XXIV

135. *Two parallelograms are equal if two sides and the included angle of one are respectively equal to two sides and the included angle of the other.*

Given ▱s $ABCD$ and $HEFG$ with AD, AB, and $\angle A$ respectively equal to HG, HE, and $\angle H$.

To prove ▱ $ABCD =$ ▱$HEFG$.

Proof. Draw diagonals DB and GE.

Prove $\triangle ADB = \triangle HGE$.

Then use exercise 39, 1.

Theorem XXV

136. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Draw quadrilateral $ABCD$ having $AB = DC$ and $AD = BC$.

Draw BD and prove the triangles formed are equal.

Then use § 125.

EXERCISE 40

1. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

2. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

3. Two rectangles are equal when two adjacent sides of one are respectively equal to two adjacent sides of the other.

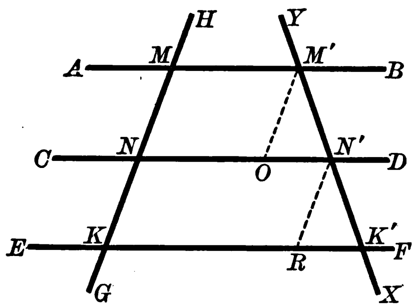
4. The diagonals of a square bisect its angles.

5. If the diagonals of a rectangle bisect its angles, the rectangle is equilateral.

Theorem XXVI

137. *If a series of parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.*

Given ∥s AB, CD, and EF intercepting equal parts on line GH.



To prove AB, CD, and EF intercept equal parts on every transversal XY.

Proof. 1. Let HG cut AB, CD, and EF at M, N, K, respectively, and XY cut these ∥s at M', N', K', respectively. Draw $M'O$ and $N'R$ respectively ∥ to MK.

2. $MN = M'O = NK = N'R$. (?)

3. Now prove $\triangle M'ON' = \triangle N'RK'$.

EXERCISE 41

1. A line which bisects one side of a triangle and is parallel to another side bisects the third side. (In § 137, if $M'K'$ were drawn through M, we would have a triangle MKK' with N the middle point on one side.)

2. The line parallel to a base of a trapezoid, and bisecting one of the non-parallel sides, bisects the other non-parallel side. (In § 137 $KK'M'M$ is a trapezoid.)

THEOREM XXVII

138. *The line joining the middle points of two sides of a triangle is parallel to the third side and equal to one half of it.*

Draw $\triangle$ *ABC*. Through *M* and *M'*, middle points of *AB* and *AC*, respectively, draw *MM'*. We then have

Given $\triangle$ *ABC* with *MM'* connecting the middle points of the two sides.

To prove $MM' \parallel BC$.

Proof. 1. Through *M* draw a line parallel to *BC*.

2. This line will bisect *AC* and must therefore pass through *M'*. Why?

3. The parallel drawn will coincide with *MM'*, therefore $MM' \parallel BC$.

4. Draw $M'X \parallel AB$ meeting *BC* at *X*.

5. Prove $\triangle AMM' = \triangle M'XC$.

6. Then, $MM' = XC = \frac{1}{2} BC$.

THEOREM XXVIII

139. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one half of their sum.*

Draw trapezoid *ABCD*, *AB* and *DC* being the parallel sides. Draw *EF* joining *E* and *F*, the middle points of *AD* and *BC*, respectively. We then have

Given trapezoid *ABCD*, with *EF* joining the middle points of the non-parallel sides.

To prove $EF \parallel AB$ and *DC*, also $EF = \frac{1}{2}(AB + DC)$.

Proof. 1. A line through $E \parallel AB$ will pass through *F*. (Exercise 41, 2.)

2. Then $EF \parallel AB$ and *DC*.

3. Draw *DB*, intersecting *EF* at *M*.

4. In $\triangle$ *ABD*, *EM* is drawn through *E* the middle point of *AD* and parallel to *AB*. (Use exercise 41, 1.)

5. Use $\triangle$ *BCD* in a similar manner.

THEOREM XXIX

140. *The bisectors of two of the angles of a triangle intersect on the bisector of the third angle.*

Draw $\triangle ABC$, also BM bisecting $\angle B$ and CM bisecting $\angle C$. We now have

Given $\triangle ABC$, and BM and CM bisecting $\measuredangle s$ B and C, respectively.

To prove that M lies in the bisector of $\angle A$.

Proof. 1. Draw AM.

2. Since BM bisects $\angle B$, M is equally distant from AB and BC. (§ 130.)

3. Since CM bisects $\angle C$, M is equally distant from BC and CA. (§ 130.)

4. Therefore M is equally distant from AB and AC.

5. Then M lies in the bisector of $\angle A$, and AM is the bisector. (Exercise 38, 1.)

141. From theorem XXIX we may assume that the three bisectors of the angles of a triangle meet in a point, and the point of intersection is equally distant from the three sides.

EXERCISE 42

1. In a parallelogram $ABCD$, M, the middle point of AB, is joined to D, and M', the middle point of DC, is joined to B. Show that the diagonal AC is trisected.

2. Show that if the middle points of the sides of a quadrilateral are joined in order, the figure formed is a parallelogram.

3. If the middle points of the sides of a rectangle are joined in order, the figure formed in a rhombus.

4. The figure formed by joining the middle points of the sides of a square taken in order is a square.

5. Show that the two bisectors of the interior angles on the same side of the transversal of two parallel lines form a right angle.

6. What figure is formed by joining the middle points of the sides of an isosceles trapezoid taken in order?

7. Show that the opposite angles of an isosceles trapezoid are supplementary.

Theorem XXX

142. *The perpendiculars erected at the middle points of the sides of a triangle meet in a point which lies in the perpendicular bisector of the third side.*

Draw the perpendicular bisectors of two of the sides. Join their point of intersection to the middle point of the third side. Use § 94. Prove in a manner similar to that used in § 140.

143. We may conclude that the three perpendicular bisectors of the three sides of a triangle meet in a point equidistant from the three vertices.

Theorem XXXI

144. *The three altitudes of a triangle meet in a common point.*

Draw $\triangle ABC$, and its three altitudes x, y, z. We now have,

Given $\triangle ABC$ and the altitudes x, y, z.

To prove that x, y, and z have a common point.

Proof. 1. Through A draw $MK \parallel BC$; through B, draw $MH \parallel AC$; through C draw $HK \parallel BA$.

2 $x \perp MK$. (§ 121.)

3. Since $MBCA$ and $ABCK$ are parallelograms, $BC = MA = AK$.

4. Hence x is the perpendicular bisector of MK.

5. Similarly y is the perpendicular bisector of HK and z of MH.

6. But the perpendicular bisectors of the sides of $\triangle MHK$ meet in a point. (143.)

7. But x, y, z are also the altitudes of $\triangle ABC$.

8. Therefore the altitudes of a triangle meet in a common point.

THEOREM XXXII

145. *Two medians* of a triangle meet in a point of the third median.*

Draw $\triangle ABC$, and medians CM and AK meeting at O. Also draw BL through O, meeting AC at L. We now have

Given $\triangle ABC$ with medians AK and CM meeting at O.

To prove that O is in the median drawn to the third side.

Proof. 1. Draw CH parallel to AK and meeting BL produced at H. Draw AH.

2. In $\triangle HBC$, $OK \parallel HC$ and bisects BC. It therefore bisects BH. (?)

3. Since O is the middle point of HB, and M the middle point of AB, $MO \parallel AH$. (?)

4. Then $AOCH$ is a parallelogram, and L is the middle point of AC. Why?

5. Hence BL is a median and O lies in the median.

146. In § 145, since $HO = OB$ and $LO = \frac{1}{2} HO$, O is $\frac{2}{3}$ the distance from B to L. Similarly we may show that O is $\frac{2}{3}$ the distance from C to M, and from A to X. Or the medians meet in a point $\frac{2}{3}$ the distance, along the median, from the vertex to the opposite side.

Polygons

147. A **polygon** is a portion of a plane bounded by three or more straight lines.

The bounding lines of a polygon are its **sides**.

Any two adjacent sides form an **angle of the polygon.**

A polygon is named with reference to its number of sides, or angles. Thus, triangle, quadrilateral, pentagon, hexagon, octagon, etc.

* *A median* is a line drawn from a vertex of a triangle to the middle point *of the opposite side.*

If the sides are equal, the polygon is **equilateral.**

If the angles are equal, the polygon is **equiangular.**

A **regular polygon** is both *equiangular* and *equilateral.*

148. Two polygons are mutually equilateral if their corresponding sides are equal.

Two polygons are mutually equiangular if their corresponding angles are equal, that is, if their angles taken in the same order are respectively equal.

Two polygons are equal if they are both mutually equilateral and mutually equiangular. They are also equal if they can be separated into the same number of triangles, equal each to each, and similarly placed.

THEOREM XXXIII

149. *The sum of the angles of any polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

Draw polygon $ABCDE$, etc. We have

Given polygon $ABCD$, etc., having n sides.

To prove $\angle A + \angle B + \angle C + \angle D + \cdots$ etc., $= 2n$ rt. $\angle s - 4$ rt. $\angle s$.

Proof. 1. From any point P within the polygon draw a line to each vertex.

2. It is evident that n triangles are formed, one for each side of the polygon.

3. The sum of the angles of each triangle is two right angles, and the sum of the angles of all the triangles together is $2n$ right angles.

4. The sum of the angles of all the triangles includes the sum of the angles of the polygon, and the angles around P which do not belong to the polygon.

5. Subtracting the sum of the angles at P from the sum of the angles of the n triangles, we have $2n$ rt. $\angle s - 4$ rt. $\angle s$ for the angles of the polygon. (§ 76, d.)

Theorem XXXIV

150. *The sum of the exterior angles of a polygon formed by producing one side at each vertex of a polygon is equal to four right angles.*

Given a polygon with exterior angles b, d, f, etc.

To prove that the sum of these angles is equal to 4 rt. $\angle$s.

Proof. 1. Let a be the interior angle of the polygon adjacent to b.

2. Then $a + b = 2$ rt. $\angle$s.

3. This same sum exists at each of the n vertices, therefore the sum of the interior and the exterior angles of the polygon is $2n$ rt. $\angle$s.

4. But the sum of the interior is $2n$ rt. $\angle$s $-$ 4 rt. $\angle$s. (?)

5. Subtracting the sum of the interior angles from the sum of the interior and exterior angles, we have

$$2n \text{ rt. } \angle\text{s} - (2n \text{ rt. } \angle\text{s} - 4 \text{ rt. } \angle\text{s}),$$

or

$$4 \text{ rt. } \angle\text{s}.$$

EXERCISE 43

1. Prove Theorem XXXIII by triangles formed by drawing all the diagonals from any one vertex of the polygon.

2. One angle of a regular polygon is $1\frac{1}{2}$ right angles. How many sides has the polygon? How many degrees is the sum of its angles?

3. One angle of a regular polygon is 120°. What kind of a polygon is it?

4. An exterior angle of a regular polygon is 120°. How many sides has the polygon?

5. Each exterior angle of a polygon is 30°. Find the sum of the interior angles.

6. One angle of a regular polygon is $\frac{1}{5}$ rt. $\angle$. Find the *number of sides.* Any trouble here? Why?

7. From one vertex of a regular polygon 5 diagonals can be drawn. Find the sum of the angles of the polygon.

8. What three lines belonging to any triangle meet in a point? Is there more than one answer to this question?

9. One exterior angle of a regular polygon is 180°. How many sides has the polygon? What is the greatest exterior angle a regular polygon can have? What is the least exterior angle possible?

151. Develop the following synopses:

Two lines are equal, if
Two angles are equal, if
Two lines are parallel, if
A quadrilateral is a parallelogram, if

Supplemental Applied Mathematics

1. A hexagonal water tank is $3' - 6''$ on a side. Find the area of the cover.

2. In nuts and heads of bolts, the distance across the flats (the distance between the parallel sides) is $\frac{3}{2}$ the diameter of the bolt plus $\frac{1}{8}''$. The thickness of the head is $\frac{1}{2}$ the distance across the flats. Find the distance across the flats and the thickness of the head on a one-inch bolt having a square head.

3. A hexagonal head is $\frac{5}{8}''$ on a side. Find the distance across the flats and the thickness of the head.

4. A square head $\frac{5}{16}''$ thick is to be milled on a cylindrical blank. Find the diameter of the blank.

5. In example 3, find the approximate diameter of the bolt.

6. A half-inch bolt has a square head. Find the distance across the flats, the thickness of the head and the diagonal of the head.

7. A one-inch steel bolt is 4″ long under the head. The head is square. Find weight of these bolts per hundred.

8. A square head measures $\frac{5}{4}$″ across the flats. The bolt is 5″ long under the head. Find weight per hundred.

9. The thickness of a hexagonal steel head is $\frac{5}{8}$″. Find weight of heads per hundred. What size bolt would you use with such head?

10. A hexagonal head is $\frac{3}{4}$″ on a side. Find the diameter of the bolt.

11. A hole is 0.185″. Could you use a $\frac{3}{16}$″ bolt in this hole? Would a $\frac{1}{4}$″ bolt be too large?

12. A hole is 1.284″. The longest diagonal of the hexagonal head used is 2.74″. Find size of bolt used, weight of nuts per hundred, number of nuts per hundred pounds. (Bolts only come in 4ths, 8ths, 16ths, 32ds, 64ths).

13. Whites of eggs coagulate at 56.6° C. Express this temperature in Fahrenheit scale. Yolks of eggs coagulate at 122° F. Express in Centigrade.

14. When eggs are made into omelets, 1 tablespoonful of milk and $\frac{1}{2}$ teaspoonful of butter are added to each egg. 7 eggs will make 5 portions for serving. Find difference in cost of 6 eggs and enough omelet to serve six persons, when eggs cost 30¢ per dozen, milk 6¢ per quart, butter 37¢ per pound. One cup of milk measures 16 tablespoonfuls, and one half pound of butter 16 teaspoonfuls.

15. 87 % of milk is water. 1 cup of milk weighs $8\frac{7}{8}$ ounces. Find weight of water contained in one quart of milk.

16. 3.3 % of milk is protein. $\frac{5}{6}$ of the protein is casein and $\frac{1}{6}$ albumen. What is the per cent of casein and albumen in milk?

17. The edible portion of cooked eggs contains 13.2 % protein. *How much* milk (liquid measure) does it take to contain *as much protein* as is in one dozen eggs?

18. A can of condensed milk costing 5¢ contains $\frac{2}{8}$ of a cup. Which is cheaper — fresh or condensed milk — if fresh milk costs 6¢ per quart? (Dilute condensed milk one third.)

19. American cheese contains 28.8% protein. How many eggs contain as much protein as 1 pound of cheese? How much milk (liquid measure) in one pound of cheese? How much cheese contains as much protein as 2 eggs?

20. Find difference in cost of 1 dozen eggs and as much cheese as would contain the same amount of protein, cheese costing 20¢ per pound and eggs 30¢ per dozen.

21. One quart of sour milk makes 1 cup of cottage cheese which weighs 6 ounces. Find difference in cost of 1 pound cottage and 1 pound American cheese. (Prices same as above.)

22. Cottage cheese contains 20.9% protein. How much cottage cheese contains as much protein as 1 pound American cheese? How much sour milk will be required to make this quantity of cottage cheese?

23. $\frac{1}{2}$ pound macaroni or 1 cup rice can be used with cheese. 1 cup rice weighs $7\frac{1}{2}$ ounces, and costs 10¢ per pound. Macaroni costs 15¢ per pound package. Find difference in cost.

24. Boiled rice contains 24.4% carbohydrates; cooked macaroni contains 15.8% carbohydrates. How many quarts cooked macaroni will contain as much carbohydrates as 5 cups cooked rice?

25. Uncooked macaroni contains 74.1% carbohydrates; cooked macaroni contains 15.8% carbohydrates. Find weight of loss in carbohydrates from cooking 1 pound macaroni.

26. I wish to make a dusting cap 18″ in diameter. How much lace is needed to put around the edge, allowing one half extra for fullness?

27. Two inches in from the edge of the cap in Example 26 I sew beading. How much beading must I buy?

CHAPTER VIII

Products and Factors

152. In exercises 5 and 14 we found the factors of monomial products. We shall now extend factoring to include the products found in exercises 27 and 29.

The Difference of Two Squares

153. TYPE I. *Multiply* $a + b$ by $a - b$. (That is, multiply the *sum* of two numbers by the *difference* of the *same* two numbers.)

By actual multiplication:

$$\begin{array}{l} a + b \\ \underline{a - b } \\ a^2 + ab \\ \underline{ - ab - b^2} \\ a^2 - b^2 \end{array}$$

That is, $(a + b)(a - b) = a^2 - b^2$.

Or, stated in words:

The product of the sum and difference of two numbers is equal to the difference of their squares.

Then to multiply the sum of two numbers, as $3x + 5$, by the difference of the same two numbers, $3x - 5$, one needs only to square the first number, $3x$, and subtract the square of the second number, 5, from it, giving for the product, $9x^2 - 25$.

EXERCISE 44

Find by inspection:

1. $(x + y)(x - y)$.
2. $(x + 3)(x - 3)$.
3. $(m + x)(m - x)$.
4. $(x + 4)(x - 4)$.

5. $(2x+1)(2x-1)$.

6. $(2x+3)(2x-3)$.

7. $(5x+7)(5x-7)$.

8. $(14y+15)(14y-15)$.

9. $(17a+19b)(17a-19b)$.

10. $(16c+25d)(16c-25d)$.

11. x^2-4 is the product of what two numbers? Are these numbers binomials, trinomials, or monomials?

12. What are the factors of x^2-25.

13. Restate the rule in Type I, making it applicable for factoring such examples as example 12. Keep in mind that factoring is a process of division; division by inspection. The dividend is given. You must find the divisor and the quotient.

Factor the following:

14. a^2-25.

15. a^2-81.

16. a^2-36.

17. $25a^2-36$.

18. $16y^2-25$.

19. $16y^2-25c^2$.

20. $16y^2c^2z^2-25x^2d^2a^4$.

21. x^4-225.

22. $289-x^4$.

23. $361x^2-529y^2$.

24. $441y^4-729a^4$.

25. $64a^2-196b^2$.

26. $a^4-256b^4=(a^2+16b^2)(a^2-16b^2)$
$=(a^2+16b^2)(a+4b)(a-4b)$.

27. $16m^4-81c^4$.

28. $(49)^2-(25)^2$.

29. $(561)^2-(559)^2$.

30. $(625)^2-(576)^2$.

31. 3^8-2^8.

32. $(a+3)^2-(a-3)^2$.

33. $(x+7)^2-(x-7)^2$.

34. $(a+b)^2-(a-b)^2$.

35. $(y+21)^2-(y-21)^2$.

36. $(17x+16y)(17x-16y)=?$

37. $(21a+23b)(21a-23b)=?$

38. $(24m+19c)(24m-19c)=?$

39. $(25x+18c)(25x-18c)=?$

40. $(15c + 26d)(15c - 26d) = ?$

41. $\frac{9}{16}a^2 - \frac{25}{4}b^2$.

42. Can you factor $a^2 + 25$? Why?

The square of the sum of two numbers.

The square of the difference of two numbers.

154. Type II. *Multiply* $a + b$ *by* $a + b$. That is, multiply the sum of two numbers by the sum of the *same* two numbers.

$$\begin{array}{r} a + b \\ \underline{a + b } \\ a^2 + ab \\ \underline{+ ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

That is, $(a + b)^2 = a^2 + 2ab + b^2$.

Or, stated in words: (a **is the first number and** b **is the second number.**)

The square of the sum of two numbers is equal to the square of the first number plus twice the product of the first and second, plus the square of the second.

Similarly, $(a - b)^2 = b^2 - 2ab + b^2$.

Or, *The square of the difference between two numbers is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

EXERCISE 45

Find by inspection:

1. $(x + z)(x + z)$.
2. $(x + 3)(x + 3)$.
3. $(a + x)^2$.
4. $(a - x)^2$.
5. $(a + 2x)^2$.
6. $(a - 2x)^2$.
7. $(5a + 3)^2$.
8. $(5a - 3)^2$.
9. $(5a + 4b)^2$.
10. $(6y + 5c)^2$.
11. $(7x - 4z)^2$.
12. $(8c - 7d)^2$.
13. $(9m + 7x)^2$.
14. $(18a - 15b)^2$.
15. $(5m - a)^2$.
16. $(6a - 5b)^2$.
17. $(5b - 6a)^2$.

18. From what do you get this product: $x^2 + 6x + 9$?

19. From what factors do you get $a^2 - 8a + 16$?

20. What are the factors of $x^2 - 10x + 25$?

21. How can you distinguish a trinomial square?

Find the factors of the following:

22. $a^2 - 6a + 9.$

23. $a^2 - 4a + 4.$

24. $4a^2 + 4a + 1.$

25. $9x^2 - 6x + 1.$

26. $9x^2 + 30x + 25.$

27. $m^2 + 12m + 36.$

28. $4c^2 + 12c + 9.$

29. $49d^2 - 14cd + c^2.$

30. $m^2k^2 + 4mkc + 4c^2.$

31. $a^4 - 18a^2 + 81 = (a^2 - 9)(a^2 - 9) = (a + 3)(a - 3)(a + 3)(a - 3)$ by Type I.

32. $x^4 - 8x^2 + 16.$

33. $16c^4 - 72c^2d^2 + 81d^4.$

34. The area of a square is $25a^2 + 40ab + 16b^2$; find one side of the square. Find the dimensions when $a = 5$, $b = -3$.

35. One side of a square is $2x + 5y$. What is the area? Find the area when $x = 6$, $y = 1$.

36. Is $a^2 + 8a + 25$ a square? Why?

37. Is $a^2 + 10a + 24$ a square? Why?

38. Is $a^2 + 10a + 25$ a square? Why?

39. Draw a square whose side is $a + b$ inches. Draw perpendiculars at the ends of a and b and show that this square is made up of the square on line a, plus two rectangles each of whose areas is ab, and a square on line b.

40. Factor $a^4 - 16$. Are your factors factorable?

41. Factor $x^4 - 81$. What type are you using?

42.
$$\begin{aligned}(81)^2 &= (80 + 1)^2\\ &= 6400 + 160 + 1\\ &= 6561.\end{aligned}$$

Write the following squares by inspection:

43. $(41)^2$ **45.** $(79)^2$ **47.** $(145)^2$ **49.** $(162)^2$

44. (39) **46.** $(132)^2$ **48.** $(153)^2$ **50.** $(169)^2$

Trinomials of the form $x^2 + kx + c$. Binomial Factors having One Term Common.

155. TYPE III. *Multiply* $x + 5$ *by* $x + 3$.

$$\begin{array}{l} x + 5 \\ \underline{x + 3} \\ x^2 + 5x \\ \underline{\quad + 3x + 15} \\ x^2 + 8x + 15 \end{array}$$

Note that in this trinomial *the first term is the square of the common term* (x). *The coefficient of* x *is the sum of the unlike terms,* 5 *and* 3. *The third term is the product of the unlike terms.*

Ex. Find the product of $4x + 3$ and $4x - 9$.

By the rule: $(4x + 3)(4x - 9) = (4x)^2 + (3 - 9)(4x) - 27.$
$= 16x^2 - 24x - 27.$

EXERCISE 46

Find by inspection:

1. $(a + 3)(a + 4)$.

2. $(c + 5)(c + 7)$.

3. $(2x + 5)(2x + 3)$.

4. $(a + 4)(a - 3)$.

5. $(a - 4)(a + 3)$.

6. $(5a - 4)(5a - 4)$.

7. $(m + 18)(m - 15)$.

8. $(k - 18)(k + 16)$.

9. $(3k + 11)(3k - 4)$.

10. $(c + 21)(c + 22)$.

11. $(b - 14)(b + 16)$.

12. $(c + 115)(c - 12)$.

13. $(d + 15)(d + 15)$.

14. $(2x - 14)(2x + 9)$.

15. $(2x + 14)(2x + 9)$.

16. $(2x + 14)(2x - 9)$.

17. $(y + 5)(y - 40)$.

18. $(z + 27)(z - 5)$.

19. $(k + 28)(k - 7)$.

20. $(a - 17)(a - 15)$.

21. From what factors do you get $x^2 + 7x + 12$?

22. Factor $x^2 - x - 12$.

To get the second terms of these binomials: *factor the third term into two factors whose sum is the coefficient of the unknown in the second term.*

Factor the following:

23. $x^2 - 2x - 8$.

24. $x^2 + 6x + 8$.

25. $x^2 - 6x + 8$.

26. $x^2 + 2x - 8$.

27. $a^2 + 9a + 20$.

28. $a^2 - a - 20$.

29. $z^2 - z - 72$.

30. $c^2 + 16c + 15$.

31. $m^2 + 8m + 16$.

32. $y^2 - 12y + 11$.

33. $y^2 - 9y - 36$.

34. $a^2 - 16a + 64$.

35. $4a^2 + 16a + 15$.
(2 a is the term common to each binomial.)

36. $9a^2 - 15a - 14$.

37. $25x^2 + 50x + 21$.

38. $c^4 - 13c^2 + 36$.

39. $m^4 - 29m^2 + 100$.

40. $x^2 + 16x + 64$.

41. $x^2 - 10x + 25$.

42. $(a+3)(a-3)(a+5)(a-5) = ?$

43. $a^2 + 3a + 2$.

44. $a^2 - 3a + 2$.

45. $a^2 + 2a + 2$.

46. $a^2 - 2a + 2$.

Polynomials having a Factor Common to Each Term

156. Type IV. *Multiply* $\mathbf{3a + 4b + 2c}$ *by* $\mathbf{5x}$.

$$\begin{array}{l} 3a + 4b + 2c \\ \underline{5x } \\ 15ax + 20bx + 10cx \end{array}$$

Each term of the product contains the common factor $5x$. This is a very important type of factoring. The first step to take in all examples is to see if the example belongs to Type IV.

Ex. 1. Factor $4a^2x - 16b^2x$.

Each term *contains* the factor $4x$.

Divide by $4x$. $$4x\overline{)4a^2x-16b^2x}$$ $$a^2-4b^2$$

The divisor is one factor, the quotient the other factor, or

$$4a^2x - 16b^2x = 4x(a^2 - 4b^2).$$

But, $a^2 - 4b^2$ can be factored by Type I.

Hence, $$4a^2x - 16b^2x = 4x(a + 2b)(a - 2b).$$

This is similar to factoring such a number as 105.

We remove a factor 3, then have

$$105 = 3(35).$$

Factoring 35,

$$105 = 3 \cdot 5 \cdot 7.$$

Ex. 2. Factor $(2a+b)x^2 - (2a+b)8x + (2a+b)15$.

$$2a+b\overline{)(2a+b)x^2-(2a+b)8x+(2a+b)15}$$ $$x^2 - 8x + 15$$

Use Type III on the quotient. Then,

$$(2a+b)x^2 - (2a+b)8x + (2a+b)15 = (2a+b)(x-3)(x-5).$$

EXERCISE 47

Factor the following:

(Be sure that you cannot still further factor your result. Check each answer.)

1. $3x^2 + 27x$.
2. $a^2b + ab^2$.
3. $5a + 25$.
4. $a^3 - ab^2$.
5. $5x^2 - 20$.
6. $(a^2 - b^2)x^2 - (a^2 - b^2)y^2$.
7. $c^2(2m+5) - c(2m+5) - 12(2m+5)$.
8. $(2c+7)4c^2 + (2c+7)20c + (2c+7)21$.
9. $x(x+3) + y(x+3)$.
10. $ax + bx + ay + by$.

Factor the first two terms and the last two terms separately.

$$x(a+b) + y(a+b).$$

The example is now like example 9.

11. $x^3 - 3x^2 - 3x + 9$. (x^2 is a factor of the first two terms *and* -3 *a factor* of the last two terms.)

12. Factor $(a+b)a^2+(a+b)2ab+(a+b)b^2$.
Compare example 9, exercise 27.

13. $x^2(2x-1)-x(2x-1)-12(2x-1)$.
What Types did you use?

14. $a^4(a^2-b^2)-b^4(a^2-b^2)$.

15. $(2m+5)m^2+(2m+5)8m+(2m+5)16$.

16. $(2m+5)m^2-(2m+5)17m+(2m+5)16$.

17. $(2m+5)m^2-(2m+5)15m-(2m+5)16$.

18. $25x^2(2x+5)-20x(2x+5)+4(2x+5)$.

19. $4m^3+12m^2+36m$.

20. $4m^3+24m^2+36m$.

21. Is $a^2+9a+25$ a square? Why?

22. $x^5-12x^4+36x^3$.

23. $(2x+1)4x^2+(2x+1)4x+2x+1$.

24. $(a^2-a-72)(2a^2-18)$.

25. $(x+3)^2-(x-3)^2$.

26. $(x+3)^2x^4-(x+3)^2 81$.

27. $x^3+3x^2-9x-27$.

28. $x^3+3x^2-6x-18$.

29. $8x^3+12x^2-10x-15$.

30. $x^3+4x^2-16x-64$.

31. $m^3-4m+7m^2-28$.

32. $12a^3+8a^2b-27ab^2-18b^3$.

Trinomials of the Form ax^2+bx+c

157. Type V. *Find the product of* $2x+5$ *and* $3x+7$.

$$\begin{array}{l} 2x+5 \\ \underline{3x+7} \\ 2\cdot3\cdot x^2+3\cdot5\cdot x+2\cdot7\cdot x+5\cdot7 \\ =2\cdot3\cdot x^2+(3\cdot5+2\cdot7)x+5\cdot7 \end{array}$$

Notice that if the first and third terms of this trinominal are combined by multiplication, the product $2\cdot3\cdot5\cdot7\cdot x^2$, comprises all the factors which make up the middle term of the product. Note also that the middle term, $29x$, is the sum of the cross products, that is, the sum of $2x\cdot7$ and $3x\cdot5$.

Reversing this multiplication process we may factor trinominals of this type.

Ex. Factor $6x^2 + 29x + 35$.

1. Find the product of the first and third terms, or $210x^2$.

2. Factor $210x^2$ so that the sum of the factors is the middle term $29x$; these factors are $14x$ and $15x$.

3. Write the trinominal in the form of a quadrinominal, using $14x$ and $15x$ for the middle terms:

$$6x^2 + 29x + 35 = 6x^2 + 14x + 15x + 35.$$

4. Now use Type IV.

$$2x(3x + 7) + 5(3x + 7) = (3x + 7)(2x + 5).$$

Hence, $$6x^2 + 29x + 35 = (3x + 7)(2x + 5).$$

It does not matter whether $14x$ or the $15x$ is connected with $6x^2$.

EXERCISE 48

Find by inspection:

1. $(2x+7)(3x+4)$.
2. $(5x+2)(2x+4)$.
3. $(6x+5)(2x-3)$.
4. $(5x+3)(3x-4)$.
5. $(2x-7)(3x-4)$.
6. $(2x-7)(3x+4)$.
7. $(2x+7)(3x-4)$.
8. $(5x+2)(2x-4)$.
9. $(7x+3)(5x-8)$.
10. $(10x-18)(7x+15)$.
11. $(8x+17)(7x-6)$.
12. $(4x+15)(9x-14)$.
13. $(6x-21)(5x+18)$.
14. $(7x+15)(7x+15)$.
15. $(13x-1)(13x+2)$.
16. $(14x+13)(13x+14)$.
17. $(14x-8)(14x+8)$.
18. $(3x-75)(2x-4)$.

Factor the following: 19. $6x^2 + 23x + 20$.

$$6x^2 \cdot 20 = 120x^2.$$
$$120x^2 = 15x \cdot 8x.$$
$$15x + 8x = 23x.$$

Then, $$6x^2 + 23x + 120 = 6x^2 + 15x + 8x + 20.$$
$$= 3x(2x + 5) + 4(2x + 5).$$
$$= (2x + 5)(3x + 4).$$

20. $6x^2+22x+20$.
21. $8x^2+22x+15$.
22. $10x^2+19x+6$.
23. $12x^2-23x+10$.
24. $8x^2+2x-15$.
25. $15x^2+23x-28$.
26. $15x^2+47x+88$.
27. $56x^2-17x-3$.
28. $4d^2+12d+9$.
29. $4x^2-14x-98$.
30. $x^2+18x+81$.
31. $36x^2+60x+25$.
32. $15x^2+14x-16$.
33. $15b^2-14b-16$.
34. $15x^2+34x-16$.
35. $10x^2-x-24$.
36. $10x^2-29x+10$.
37. $14x^2+53x+14$.
38. $(2x-3)8x^2+(2x-3)22x+(2x-3)15$.
39. $8x^4+2x^3-15x^2$.
40. $60x^3-115x^2+50$.
41. $(a^2-9)4a^2-(a^2-9)4a+a^2-9$.
42. $4m^2(2m+5)+12m(2m+5)+9(2m+5)$.
43. $2y^2+y-28$.
44. $5c^3-11c^2-36c$.
45. $24m^3+14m^2-3m$.

Binomials having Both Terms Raised to the Same Power

158. TYPE VI. Such types arise from the following products: By actual multiplication

$$(a^2+ab+b^2)(a-b)=a^3-b^3.$$

$$\begin{array}{l}
a^2+ab+b^2 \\
\underline{a-b\qquad\qquad} \\
a^3+a^2b+ab^2 \\
\underline{\qquad -a^2b-ab^2-b^3} \\
a^3 \qquad\qquad\qquad -b^3.
\end{array}$$

That is,
$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

Note the form of the quotient: we shall speak of a as the leading letter and b as the following letter.

1. The first term of the quotient contains the leading letter raised to a power *one* less than its power in the dividend.

2. The power of the leading letter becomes less by *one* in each succeeding term of the quotient.

3. The following letter appears in the second term of the quotient and its power increases by *one* in each succeeding term.

4. The signs of the quotient are all $+$.

Similarly, $$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3,$$

and $$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

Ex. 1. $$\frac{x^3 - 64}{x - 4} = x^2 + x \cdot 4 + (4)^2.$$
$$= x^2 + 4x + 16.$$

Here, a is x,
and b is 4.

Likewise, $$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$$

and $$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Note that when the binomial divisior is all positive, the terms of the quotient are alternately $+$ and $-$. Do not divide the sum of the same *even* powers by the sum of the roots.

EXERCISE 49

Factor the following:

1. $x^3 - y^3$.
2. $x^3 + y^3$.
3. $x^3 - 8$.
4. $x^3 + 8$.
5. $x^3 - 27$.
6. $x^3 + 27$.
7. $8x^3 + 27$. Here $a = 2x$ and $b = 3$.
8. $8x^3 - 27$.
9. $125a^3 + 1$.

10. $125a^3-8$.
11. $64m^3+27$.
12. $64m^3-y^3$.
13. $64m^3+27y^3$.
14. $8c^3-216d^3$.
15. x^5-32.
16. x^5+32.
17. a^5+243.
18. m^3-216.
19. b^3+512c^3.
20. $125y^3-729d^3$.
21. x^6+27 (Regard x^6 as a cube.)
22. x^6+64.
23. a^6+216.

Trinomials which may be referred to Type I by the addition and subtraction of some number.

159. TYPE VII. $x^4+bx^2+c^2$. Usually in this type one term is a fourth power, one is a square, and one is the product of a square and some number.

Ex. Factor x^4+2x^2+9.

If the middle term were $6x^2$, this trinomial would belong to Type II. Adding and subtracting $4x^2$, we have.

$$x^4+2x^2+9=x^4+6x^2+9-4x^2.$$
$$=(x^2+3)^2-(2x)^2.$$

By Type I,
$$=(x^2+3+2x)(x^2+3-2x).$$

EXERCISE 50

Factor the following:

1. x^4+x^2+25.
2. $9x^4+8x^2+4$.
3. x^4+5x^2+49.
4. x^4+10x^2+49.
5. x^4-23x^2+49.
6. $a^4-39a^2b^2+49b^4$.
7. a^4+4.
8. $4y^4-37y^2+9$.
9. $4y^4+8y^2+9$.
10. $25x^4-65x^2+16$.

Fill in the term which will make each of the following a trinomial square:

11. $a^2+4a+?$
12. $4m^2+28m+?$

13. $x^2 - 24x$.

14. $c^2 + 4$.

15. $? + 6x + 9$.

16. $x^2 + 8x + ?$

17. $x^2 + 5x + ?$

18. $c^2 - 9c + ?$

19. $c^4 + 16$.

20. $9m^2 - 18m$.

21. $25a^2 - 30ab$.

22. $25a^2 - 25ab$.

23. $25a^2 - 20ab$.

24. $25a^2 - 18ab$.

160. The seven types of factorable expressions should be committed to memory. Before attempting to factor any expression, select the type to which it belongs.

Hints on Factoring

1. For all expressions:

First try Type IV.

2. Test binomials by Types I, IV, VI.

3. Test trinomials by Types II, III, V, VII.

4. Be sure that each factor will admit of no further factoring.

5. Several types may be needed to completely factor an expression, *e.g.*, example 13, exercise 47.

Types

I. $a^2 - b^2 = (a + b)(a - b)$.

II. $(a + b)(a + b) = a^2 + 2ab + b^2$.

$(a - b)(a - b) = a^2 - 2ab + b^2$.

III. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

IV. $ad + bd + cd = d(a + b + c)$.

V. $ax^2 + bx + c$.

VI. $a^n - b^n$.

$a^n + b^n$.

VII. $x^2 + bx^2 + c^2$.

EXERCISE 51

Factor the following:

1. $y^2 - 2y + 1$.
2. $y^2 - 1$.
3. $y^2 - 2y - 8$.
4. $y^2 + 2y - 8$.
5. $50x^3 + 45x^2 - 45x$.
6. $10x^2(2x - 3) + 45x(2x - 3) - 45(2x - 3)$.
7. $(341)^2 - (339)^2$.

8. One side of a square is 585′. This square is surrounded by a concrete walk of uniform width, whose outside perimeter is 2388′. Find the cost of the walk at 14¢ per square foot.

9. One square is so placed within another that a space of uniform width is between the sides of the outer and the inner squares. The sides of the squares are 76″ and 69″, respectively. How many square feet in the difference between the areas of these squares?

10. The area of a square is $36x^2 + 84x + 49$. Find one side. How large is the square when $x = -4$? If x were -9, would the area still be positive? Could you substitute a value of x that would make the area negative? Why?

11. The area of a square is $49a^2 - 28a + 4$. Find one edge when $a = \frac{1}{7}$.

12. Is $a^2 + 10a + 16$ a trinomial square? Why?

13. The area of a triangle is $\dfrac{3x^2 + 17x - 28}{2}$. Find the base and altitude. If x is 2 and the triangle is isosceles, construct the triangle. Is more than one such triangle possible? Why?

14. The area of a right triangle is $5x^2 + 4x - 12$. Construct the triangle when $x = 2$.

15. Find by inspection $(2x + 3)(2x - 3)(x + 4)(x - 4)$.

16. Factor $x^4 - 25x^2 + 144$.

17. $a^4 - 29a^2 + 100 = ?$

18. $a^4 - 81 = ?$

19. $(2m + 3)^2 - 36 = ?$

20. $(2m - 5)^2 - 25m^2 = ?$

21. $(2m - 5)^2 - (2m + 5)^2 = ?$

22. $(4x + \frac{1}{4})^2 = ?$

23. Factor $4x^2 + x + \frac{1}{4}$.

24. $\left(\frac{2x+1}{4} + \frac{4}{2x+1}\right)^2 = ?$

25. $\left(\frac{x+1}{5} + \frac{5}{x+1}\right)^2 = ?$

26. Factor $5x^2 + 11x - 36$.

27. The area of a rectangle is $x^2 + 7x + 12$. Find its dimensions when $x = 4''$.

28. Find the dimensions of a rectangle whose area is $4x^2 - 169y^2$.

29. Find the dimensions of a rectangle whose area is $225a^2 - 289b^2$.

30. Find the dimensions of a rectangle whose area is

$$x^2 + (3m + 4a)x + 12ma.$$

What are its dimensions when $x = 4''$, $m = 1''$, $a = 1''$? How does this rectangle compare with that in example 27?

31. Compare the adjacent sides in a rectangle whose area is

$$25x^2 - 40xy + 16y^2.$$

32. Factor $9x^2 + 42xcd + 49c^2d^2$.

33. Factor $4c^2 - 256c^4d^2m^2$.

34. $15x^2 + 29x - 14$.

In working examples under Type V, it is not necessary to carry out all the steps indicated in § 157.

$$(15x^2)(-14) = -210x^2,$$
$$-210x^2 = (35x)(-6x).$$

Connect either of these factors with $15x^2$; for example,

$$15\,x^2 + 35\,x.$$

Factor this expression, $5\,x(3\,x + 7)$.

Then, $(3\,x + 7)$ is one factor of $15\,x^2 + 29\,x - 14$, $5\,x$ is the first term of the other factor. The second term of the second factor is formed by dividing -14 by $+7$.

The written work would appear as follows:

Factor $15\,x^2 + 29\,x - 14$.

$$(15\,x^2)(-14) = -210\,x^2,$$
$$-210\,x^2 = (35\,x)(-6\,x),$$
$$15\,x^2 + 35\,x = (3\,x + 7)(5\,x),$$
$$15\,x^2 + 29\,x - 14 = (3\,x + 7)(5\,x - 2).$$

Factor:

35. $6\,y^2 + 14\,y - 12$.

36. $10\,m^2 + m - 21$.

37. $40\,c^2 + 7\,c - 3 = 0$.

38. $6\,x^2 + x - 126 = 0$.

39. $10\,x^2 + 51\,cx + 56\,c^2$.

40. $36\,x^2 - 181\,x + 225 = 0$.

41. $30\,x^2 - 27\,x - 21$.

42. $96\,y^2 - 24\,y - 12$.

43. $36\,x^2 - 12\,x - 120$.

44. $48\,z^2 + 50\,z + 2$.

45. $31\,a^2 - 151\,a - 20$.

46. $48\,a^2 + 128\,a - 48$.

47. $25\,a^2 + 95\,a - 20$.

48. $64\,m^2 - 40\,m + 4$.

Solutions by Factoring

161. If the product of two or more finite numbers is zero, at least one of the numbers is zero.

That is, if $(x - 3)(x - 4) = 0$, either $x - 3$ or $x - 4$ must be zero. If $x - 3 = 0$, $x = 3$. If $x - 4 = 0$, $x = 4$.

Such conditions give a method for solving equations which are higher than the first degree.

Ex. Solve $x^2 - 4\,x = 21$.

Transposing so that the second member of the equation is 0, we have,

$$x^2 - 4\,x - 21 = 0.$$

Factoring by Type III,

$$(x-7)(x+3)=0.$$

Placing the first factor equal to zero, we have,

$$x=7.$$

Placing the second factor equal to zero,

$$x=-3.$$

These should be the roots of the original equation.

Checking for $x=7$,

$$7^2-4\cdot 7-21=0,$$
$$49-28-21=0.$$

For $x=-3$,

$$(-3)^2-4(-3)-21=0.$$
$$9+12-21=0.$$

Both roots satisfy the equation.

EXERCISE 52

Find value:

1. $24(x+5)(x-3)(x+1)$, when $x=0$.
2. $16(x-1)(x+2)(x+8)$, when $x=1$.
3. $1625(x+3)(x+5)(x-16)$, when $x=-5$.

Solve by factoring. (Check each root.)

4. $x^2-7x+12=0$.
5. $y^2+5y+6=0$.
6. $y^2+7y+6=0$.
7. $x^2-81=0$.
8. $x^3-9x^2-9x+81=0$.
9. $4x^2+12x=-9$.
10. $y^3-y^2-9y+9=0$.
11. $v^2-8v=-16$.
12. $x^2-x=12$.
13. $m^2-4m=0$.
14. $48x^2-12x=0$.
15. $y^2-11y-102=0$.
16. $9z^2-30z+25=0$.
17. $16x^2+42x=-5$.
18. $(2a+7)a^2-(2a+7)4a+(2a+7)4=0$.
19. $(x^2-16)4x^2-(x^2-16)28x+(x^2-16)49=0$.
20. $(25z^2-225)(49z^2-289)=0$.
21. How many roots to a first degree equation? second *degree? third* degree? *n*th degree?

Supplemental Applied Mathematics

1. A 10-inch steel pipe is 10.19″ inside diameter, 10.75″ outside diameter. Find weight per lineal foot.

2. A 9-inch pipe is 8.937″ inside diameter and 9.625″ outside diameter. Find weight per lineal foot.

3. A steam pipe is 12.750″ outside diameter and is made of steel $\frac{3}{8}$″ thick. Find the inside diameter.

4. The inside diameter of a 2-inch pipe is 2.067″. If water is forced through this pipe at the rate of 10′ per second, how many gallons can the pipe deliver per hour?

5. A 6-inch main, inside diameter 6.065″, in the same system as the 2-inch main above would deliver how many gallons per hour? Could you use the result obtained in example 4 in solving this problem, and have a fairly accurate result?

6. A pump delivers 23.5 gallons of water per stroke and is set for 16 strokes per minute. What weight of water is delivered per hour? (Water weighs $62\frac{1}{2}$ pounds per cubic foot.)

7. A technical school needs 75,000 sheets of paper $8\frac{1}{4}'' \times 10\frac{1}{2}''$. The stock is $17'' \times 22''$, 16 pounds to the ream of 500 sheets, and costs $6\frac{3}{4}$¢ per pound. How many reams must be ordered and what is the cost? (This is a problem in mental arithmetic.)

8. 30,000 cards $5'' \times 4''$ are to be cut from stock $17'' \times 20\frac{1}{2}''$. Weight 30 pounds to ream, price 12¢ per pound. How much stock was bought? What did it cost? Does it make any difference which way the stock is cut?

9. In a fuel test 100 pounds of coke was found to contain 6.01 pounds of ash and .583 pounds of sulphur. The balance was carbon. What per cent of carbon was there?

10. Air is .001276 times as heavy as water. What is the weight of the air in your classroom?

11. If a body falls freely in space, the distance fallen is equal to $\frac{1}{2}g$ times the square of the time in seconds, where g is the force of gravity (32.15 feet).

The equation for this law is usually written

$$s = \frac{1}{2}gt^2, \text{ where } s \text{ is the distance in feet.}$$

A stone dropped into a cañon is seen to strike the water at the bottom of the cañon in 8 seconds. How deep is the cañon?

12. A stone dropped over a precipice fell for 10 seconds before striking the ground. How far did it fall?

13. An aeroplane was sailing 1000 feet above the ground when one of the passengers dropped a handbag overboard. How long did it take the handbag to reach the ground?

14. 56 hurdles 5 feet long just reach across a field. How many hurdles 4 feet long would be needed?

15. For bronze bearings the Pennsylvania Railroad uses the following alloy, 77 % copper, 8 % tin, 15 % lead. How many pounds of copper, tin and lead are used in making 900 pounds of bearings?

16. In one lot of 402 castings, 24 were spoiled, in a second lot of 500 castings, 38 were spoiled, in a third lot of 321, 22 were spoiled. In which lot was the largest percentage of loss due to spoiled castings?

17. In testing our shop drive, 9.24 horse-power went to the lathes and .75 horse-power went to the grindstone. The motor delivered 11.2 horse-power. What per cent of the power went to the belting and shafts?

18. My competitor and I handle hardware. For the same set of articles my competitors' prices are \$ 3, \$ 3.30, \$ 3.55 and \$ 3.70. His trade discounts are 25 %, $7\frac{1}{2}$, 5 and 2. My list prices for the same set are \$ 6.10, \$ 6.70, \$ 7.20 and \$ 7.50 and my trade discount is 60 %, $7\frac{1}{2}$, $7\frac{1}{2}$, 5 and 2. In making a bid *how do the net* prices compare?

19. Three men can set up a line shaft in 8 days. How many men can set up the same shaft in 3 days?

20. Fourteen men are at work installing the machinery in a shop. They work for 8 days and finish half the work. The work must be completed in 5 more days. How many men must be added?

21. 200 men were completing the work on the Technical High School. The job had to be completed Oct. 5. On Oct. 1, the contractor found that there were still 500 days' work to be done. How many men could he lay off and finish the job on time?

22. One cup ground coffee makes 6 cups boiled. $\frac{3}{4}$ cup boiled coffee serves 1 person. How many level tablespoonfuls of ground coffee should be used for each person?

23. If coffee costs 35¢ per pound and there are $4\frac{1}{8}$ cups per pound, find cost of enough beverage for one person.

24. For filtered coffee $\frac{2}{3}$ cup is used with 5 cups water. Find difference in cost of enough boiled and filtered coffee to serve 6 portions.

25. To clear 1 quart coffee, $\frac{1}{8}$ white of egg or several egg shells may be used. It takes 11 yolks or 9 whites of eggs to measure 1 cup. Assuming that the yolks can be used for other purposes, eggs selling at 30¢ per dozen, and 1 quart coffee used daily, how much can be saved in a month by using egg shells to clear the coffee?

26. Tea made from Ceylon tea leaves contained 8.6% tannic acid after five minutes' infusion; 10.88% tannic acid after thirty minutes' infusion. Find the difference in the quantity of tannic acid extracted from tea leaves steeped for 5 and 10 minutes during a month, if $\frac{1}{2}$ pint tea is used daily and 1 pint of the beverage weighs 1 pound.

27. A tea contained 6.8% tannic acid after five minutes' infusion and 16.3% tannic acid after forty minutes' infusion. Find the difference in the quantity of tannic acid extracted

from the tea leaves after steeping 5 and 40 minutes du: a month if $\frac{3}{8}$ pint tea is used daily.

28. Green tea leaves contain 10.64 % tannic acid; black leaves contain 4.89 % tannic acid; 1 cup tea leaves weigl ounces. If 1 teaspoonful tea leaves is used in making a cu] tea each day, find the difference in the quantities of taı acid extracted from black and green tea during a month. cup tea leaves measures 48 teaspoonfuls.

CHAPTER IX

Fractions

162. A **fraction** is an indicated division. It is written in the form $\frac{a}{b}$, the number above the line being the numerator or dividend, the number below the line being the denominator or divisor.

A fraction may be positive or negative (see Chapter III), the sign $+$ indicating that the quotient is to be added, the sign $-$ indicating that the quotient is to be subtracted.

Thus, $-\frac{-4}{2}$ means that the quotient arising from $-4 \div 2$ must be subtracted, the result is $+2$.

We see that three signs are involved in every fraction. The sign of the numerator, the sign of the denominator, and the sign of the fraction. Any two of these signs may be changed without changing the value of the fraction.

Thus, $$+\frac{4}{2} = -\frac{-4}{2} = \frac{-4}{-2} = -\frac{4}{-2}.$$

And $$\frac{a-3}{a+6} = -\frac{a-3}{-a-6} = \frac{-a+3}{-a-6} = -\frac{-a+3}{a+6} = -\frac{3-a}{a+6}.$$

Such changes often simplify operations with fractions.

Principles of Fractions

163. The following principles govern operations in fractions:

1. *Multiplying the numerator of a fraction by a number multiplies the fraction.*

This depends on axiom 3, § 23,

$$\frac{D}{d} = Q.$$

Multiply both sides of the equation by some number, 5, we have,

$$5\left(\frac{D}{d}\right) = 5\,Q,$$

or

$$\frac{5\,D}{d} = 5\,Q.$$

2. *Multiplying the denominator of a fraction by a number divides the fraction.*

The pupil may show that this principle depends on axiom 3.

3. *Multiplying both numerator and denominator of a fraction by the same number does not change its value.* Why? Axioms?

4. *Dividing the numerator of a fraction by a number divides the fraction.*

$\frac{D}{d} = Q.$ Then $\frac{\frac{D}{5}}{d} = \frac{Q}{5}.$ The pupil may explain use of axioms.

5. *Dividing the denominator multiplies the fraction.*

The pupil may illustrate.

6. *Dividing both numerator and denominator by the same number does not change the value of the fraction.* Explain.

Reduction

164. Principles 3 and 6 are involved in the reduction of fractions, 3 in reduction to higher terms, 6 in reduction to lower terms.

Ex. 1. Reduce $\frac{5}{6}$ to higher terms.

$$\frac{5}{6} \cdot \frac{4}{4} = \frac{20}{24}.$$

Ex. 2. Reduce $\frac{5}{6}$ to 48ths.

$$6 = 2 \cdot 3;\ 48 = 2^4 \cdot 3$$

$$2 \cdot 3\overline{)2^4 \cdot 3}$$

$$2^3$$

Hence, multiplying both numerator and denominator by 2^3 or 8, we have,

$$\frac{5}{6}\cdot\frac{8}{8}=\frac{40}{48}.$$

Ex. 3. Reduce $\dfrac{a-3}{a+6}$ to a fraction whose denominator is a^2-a-42.

$$a^2-a-42=(a+6)(a-7).$$

$$a+6\overline{)(a+6)(a-7)}.$$
$$a-7$$

Hence, multiplying both numerator and denominator by $a-7$, we have,

$$\frac{a-3}{a+6}=\frac{(a-3)(a-7)}{(a+6)(a-7)}=\frac{a^2-10a+21}{a^2-a-42}.$$

EXERCISE 53

1. Reduce to 144ths: $\frac{5}{18}$, $\frac{1}{6}$, $\frac{7}{36}$, $\frac{9}{12}$.

2. Reduce to 512ths: $\frac{3}{16}$, $\frac{5}{32}$, $\frac{1}{64}$.

3. Reduce to 19ths: $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$.

4. Reduce to fractions whose denominator is a^2-a-12.

$$\frac{a+4}{a+3},\qquad \frac{a-7}{a-4}.$$

5. Reduce to $(4x^2+12x+9)$ths: $\dfrac{2x-3}{2x+3}$.

6. Reduce to $(4x^2-12x+9)$ths: $\dfrac{2x+3}{2x-3}$.

7. Reduce to $(x-5)(x+5)(x+1)$ths:

$$\frac{2}{x^2-25};\ \frac{4x-1}{x^2+6x+5};\ \frac{2x+7}{x^2-4x-5};\ \frac{1}{x+1}.$$

8. Reduce to $(4x-1)(2x+5)(3x-1)$ths:

$$\frac{-4}{8x^2+18x-5};\ \frac{3x+2}{12x^2-7x+1};\ -\frac{8x+5}{6x^2+13x-5};\ -\frac{-(x+2)}{12x^2-7x+1}.$$

9. Reduce to $(4x^2-9)$ ths:

$$-\frac{5x+2}{2x+3};\ \frac{2x+4}{2x+3};\ \frac{5x+2}{-2x+3}.\quad (\S\ 162.)$$

10. Reduce to $14(x^2 - x - 72)$ths:

$$\frac{8x-1}{7(x-9)};\quad \frac{-5}{2(x+8)};\quad \frac{11}{14}.$$

165. *Reduction to lower terms.*

Ex. 1. Reduce $\frac{36}{54}$ to lower terms.

$$\frac{36}{54} = \frac{2^2 \cdot 3^2}{2 \cdot 3^3}.$$

By inspection we see that $2 \cdot 3^2$ are factors common to both numerator and denominator. Dividing both numerator and denominator by $2 \cdot 3^2$ (principle 6), we have

$$\frac{36}{54} = 2 \cdot 3^2 \Big| \frac{2^2 \cdot 3^2}{2 \cdot 3^3} = \frac{2}{3}.*$$

Ex. 2. Reduce $\dfrac{a^2 - a - 12}{a^2 - 9a + 20}$ to lower terms.

$$\frac{a^2 - a - 12}{a^2 - 9a + 20} = \frac{(a-4)(a+3)}{(a-4)(a-5)}.$$

Dividing both numerator and denominator by the common factor $a-4$, we have

$$\frac{a^2 - a - 12}{a^2 - 9a + 20} = (a-4) \Big| \frac{(a-4)(a+3)}{(a-4)(a-5)} = \frac{a+3}{a-5}.$$

EXERCISE 54

Reduce to lower terms:

1. $\frac{38}{361}$. 3. $\frac{80}{144}$. 5. $\frac{36}{216}$. 7. $\frac{34}{102}$. 9. $\frac{95}{133}$.

2. $\frac{54}{289}$. 4. $\frac{64}{512}$. 6. $\frac{72}{256}$. 8. $\frac{85}{119}$. 10. $\frac{54}{324}$.

11. $\dfrac{a^2-9}{a^2-6a+9}$. 13. $\dfrac{m+1}{m^2+1}$. 15. $\dfrac{z^2+3z-10}{2z^2+3z-14}$.

12. $\dfrac{m^2+2m+1}{m^2-1}$. 14. $\dfrac{m^4-64}{m^4-16m^2+64}$. 16. $\dfrac{8c^3-27d^3}{4c^2-9d^2}$.

17. $\dfrac{4c^2+4cd-15d^2}{8c^3-27d^3}$. 18. $\dfrac{8c^3-27d^3}{12c^3+18c^2d+27cd^2}$.

* The bar | as here used indicates division of both numerator and denominator of the fraction.

19. $\dfrac{12c^3 + 12c^2d - 45cd^2}{12c^3 + 48c^2d + 45cd^2}$.

20. $\dfrac{9y^2 + 18yz - 16z^2}{18y^3 + 36y^2z - 32yz^2}$.

21. $\dfrac{9y^2 - 12yz + 4z^2}{9y^2 - 4z^2}$.

22. $\dfrac{27y^3 - 8z^3}{(18y^2 - 12yz)(9y^2z + 6yz^2 + 4z^3)}$.

23. $\dfrac{6y^2 - 13yz + 6z^2}{3y^2 + 19yz - 14z^2}$.

24. $\dfrac{(y + 7z)9y^2 - (y + 7z)12yz + (y + 7z)4z^2}{3y^2 + 19yz - 14z^2}$.

25. $\dfrac{(3y - 2z)9y^2 + 18yz(3y - 2z) - (3y - 2z)16z^2}{(27y^4 - 8yz^3)(3y - 2z)^2}$.

Multiples

166. A **common multiple** of two or more numbers is a multiple of each of them, *e.g.*, 48 is a common multiple of 12 and 16. A common multiple must contain all the factors of the numbers involved.

Ex. 1. Find a common multiple of 18 and 24.

$$18 = 2 \cdot 3^2.$$
$$24 = 2^3 \cdot 3.$$

The different factors concerned are 2 and 3.

The common multiple of these numbers must be made up of the product of 2 at least three times as a factor and 3 at least twice as a factor. That is, a number cannot contain 18 an integral number of times unless it has as factors $2 \cdot 3^2$. It cannot contain 24 an integral number of times unless it has as factors $2^3 \cdot 3$. To contain both 18 and 24 it must at least have the factors $2^3 \cdot 3^2$. The common multiples of 18 and 24 are, therefore,

$$2^3 \cdot 3^2,\quad 2^4 \cdot 3^2,\quad 2^3 \cdot 3^3,\quad 2^4 \cdot 3^3,\quad 2^3 \cdot 3^2 \cdot 5, \text{ etc.},$$

or $$72,\quad 144,\quad 216,\quad 312,\quad 360,\quad \text{etc.}$$

167. The **lowest common multiple** contains all the factors of the numbers involved the least number of times.

To find the l. c. m., form a product of all the *different* factors of the given numbers, and give to each factor the highest exponent found in any of the numbers.

Ex. 1. Find the l. c. m. of 45, 48, 54.

$$45 = 3^2 \cdot 5.$$
$$54 = 2 \cdot 3^3.$$
$$48 = 2^4 \cdot 3.$$

The l. c. m. is, therefore, $2^4 \cdot 3^2 \cdot 5.$

Ex. 2. Find the l. c. m. of $a^2 - 9$, $a^2 - 6a + 9$, $a^2 + 3a - 18$.

$$a^2 - 9 = (a+3)(a-3).$$
$$a^2 - 6a + 9 = (a-3)^2.$$
$$a^2 + 3a - 18 = (a-3)(a+6).$$

The l. c. m. is $(a-3)^2(a+3)(a+6).$

How many times will this l. c. m. contain $a^2 + 3a - 18$?

Ex. 3. Find the l. c. m. of $x - 1$, $x^2 + 9$, $x - 5$.

These numbers are all prime, their l. c. m. is their product, or

$$(x-1)(x^2+9)(x-5).$$

EXERCISE 55

Find l. c. m.:

1. $n^3 + 2n^2$, $n^3 - 4n$.
2. $a^2 - 9$, $a^2 - a - 12$.
3. 72, 48, 27.
4. 120, 18, 30.
5. 20, $x^2 - 16$, $3x^2 - 6x - 24$.
6. 15, $3x^2 - 13x - 10$, $x^2 - 25$, 12.
7. $4x^2 - 1$, $12x^2 + 12x + 3$, $4x^2 - 4x + 1$.
8. $z^3 - 27$, $z^2 + 3z + 9$, $z^2 - 9$.
9. $m^2 + 3m - 10$, $2m^2 + 7m - 15$, $2m^2 - 7m + 6$.
10. $x + 3$, $x - 3$, $x^2 - 6$.

Addition

168. Fractions, like other numbers, must be of the same denomination *before* they can be added. To reduce fractions to

the same denomination, the lowest common multiple of their denominators must be found, and principle 3 (§ 163) employed.

Ex. 1. Add $\frac{7}{24}$, $\frac{5}{18}$, $\frac{3}{16}$.

By § 167, the l. c. m. of 24, 18, 16 is $2^4 \cdot 3^2$.
In such work as reducing to l. c. d., always divide by factors.

$$2^4 \cdot 3^2 \div 24 = 2 \cdot 3.$$

Multiplying 7 and 24 by $2 \cdot 3$, we have

$$\frac{7}{24} = \frac{7 \cdot 2 \cdot 3}{24 \cdot 2 \cdot 3} = \frac{42}{144}.$$

Similarly, $$\frac{5}{18} = \frac{5 \cdot 2^3}{18 \cdot 2^3} = \frac{40}{144},$$

and $$\frac{3}{16} = \frac{3 \cdot 3^2}{16 \cdot 3^2} = \frac{27}{144}.$$

Then, $$\frac{7}{24} + \frac{5}{18} + \frac{3}{16} = \frac{42}{144} + \frac{40}{144} + \frac{27}{144} = \frac{109}{144}.$$

Ex. 2. Find the sum of

$$\frac{a+2}{a^2-25} + \frac{a}{a^2-10a+25} - \frac{2a-1}{3(a^2-2a-15)}.$$

$$a^2 - 25 = (a+5)(a-5).$$
$$a^2 - 10a + 25 = (a-5)^2.$$
$$3(a^2-2a-15) = 3(a-5)(a+3).$$
$$\text{l. c. m.} = 3(a+5)(a+3)(a-5)^2.$$
$$3(a+5)(a+3)(a-5)^2 \div (a^2-25) = 3(a+3)(a-5).$$

Then, $$\frac{a+2}{a^2-25} = \frac{3(a+2)(a+3)(a-5)}{3(a+5)(a+3)(a-5)^2}.$$

$$3(a+5)(a+3)(a-5)^2 \div (a^2-10a+25) = 3(a+5)(a+3),$$

and $$\frac{a}{a^2-10a+25} = \frac{3a(a+5)(a+3)}{3(a+5)(a+3)(a-5)^2},$$

and $$\frac{2a-1}{3(a^2-2a-15)} = \frac{(2a-1)(a+5)(a-5)}{3(a+5)(a+3)(a-5)^2}.$$

Hence, $$\frac{a+2}{a^2-25} + \frac{a}{a^2-10a+25} - \frac{2a-1}{3(a^2-2a-15)}$$

$$=\frac{3(a+2)(a+3)(a-5)}{3(a+5)(a+3)(a-5)^2}+\frac{3a(a+5)(a+3)}{3(a+5)(a+3)(a-5)^2}$$

$$-\frac{(2a-1)(a+5)(a-5)}{3(a+5)(a+3)(a-5)^2}$$

$$=\frac{3a^3-57a-90+3a^3+24a^2+45a-2a^3+a^2+50a-25}{3(a+5)(a+3)(a-5)^2}.$$

(*Note the change of sign in the last numerator. Why? Always watch for such negative numerators*).

$$=\frac{4a^3+25a^2+38a-115}{3(a+5)(a+3)(a-5)^2}.$$

Ex. 3. Find the sum of $\dfrac{1}{x-2}-\dfrac{2}{x-3}+\dfrac{3}{x-4}$.

l. c. m. $=(x-2)(x-3)(x-4)$.

Then, $\dfrac{1}{x-2}-\dfrac{2}{x-3}+\dfrac{3}{x-4}$

$$=\frac{1(x-3)(x-4)-2(x-2)(x-4)+3(x-2)(x-3)}{(x-2)(x-3)(x-4)}$$

$$=\frac{x^2-7x+12-2x^2+12x-16+3x^2-15x+6}{(x-2)(x-3)(x-4)}$$

$$=\frac{2x^2-10x+2}{(x-2)(x-3)(x-4)}.$$

EXERCISE 56

Find the following sums:

1. $\dfrac{1}{x+2}+\dfrac{1}{x+5}$.

2. $\dfrac{1}{x+2}-\dfrac{1}{x+5}$.

3. $\dfrac{x-1}{x^2-4}+\dfrac{x+1}{x+2}$.

4. $\dfrac{x+2}{x^2-9}+\dfrac{x+3}{x^2-6x+9}$.

5. $\dfrac{a}{a+b}+\dfrac{b}{a-b}$.

6. $\dfrac{c+1}{c-4}-\dfrac{c-1}{c+4}$.

7. $\dfrac{9}{c^2-25}+\dfrac{8}{c^2+5c}$.

8. $\dfrac{m}{m+1}+\dfrac{m}{m-1}+m$.

9. $\dfrac{5}{x^2-x-12}+\dfrac{5}{x^2-16}-\dfrac{5}{x^2+7x+12}$.

10. $1+\dfrac{(a-b)^2}{(a+b)^2}$. 11. $1-\dfrac{(a-b)^2}{(a+b)^2}$. 12. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

13. $\dfrac{5(x+y)}{x^2+4\,xy+3\,y^2}-\dfrac{5(x-y)}{x^2+2\,xy-3\,y^2}$.

14. $\dfrac{1}{2\,a+3}-\dfrac{1}{2\,a-3}+\dfrac{6}{4\,a^2+9}$. 16. $\dfrac{2\,x+1}{2\,x-1}-\dfrac{4+x}{1-4\,x^2}+3$.

15. $\dfrac{2\,x+1}{2\,x-1}-\dfrac{4+5\,x}{1-2\,x}$.

17. $$\frac{x^2+1}{x^2-1}+\frac{4\,x}{x+3}=\frac{x^3+3\,x^2+x+3+4\,x^2-4\,x}{(x^2-1)(x+3)}$$
$$=\frac{x^3+7\,x^2-3\,x+3}{x^3+3\,x-x-3}.$$

If the degree of the numerator is equal to or greater than that of the denominator, the fraction is *improper*, and *may* be reduced, as in arithmetic, by dividing the numerator by the denominator. Then

$$\frac{x^3+7\,x^2-3\,x+3}{x^3+3\,x^2-x-3}=1+\frac{4\,x^2-2\,x+6}{x^3+3\,x^2-x-3}.$$

Always reduce the result to its simplest form:

18. $1+\dfrac{3}{x-1}+\dfrac{4}{x^2-1}$. 20. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}$.

19. $\dfrac{a^2+1}{a^2+5}+\dfrac{a-3}{a+2}$. 21. $\dfrac{5\,y-3\,c}{25\,y^2-9}+\dfrac{5\,y+3\,c}{5\,y-3}$.

22. $\dfrac{5\,y-3\,c}{25\,y^2-9\,c^2}+\dfrac{5\,y+3\,c}{5\,y-3\,c}$.

23. $\dfrac{2\,c^2+7\,c-4}{2\,c^2+11\,c-6}-\dfrac{4\,c^2+8\,c-5}{2\,c^2+5\,c-3}$.

24. $\dfrac{3\,m^2+13\,m-10}{3\,m^2+17\,m+10}+\dfrac{3\,m^2+17\,m+10}{3\,m^2+13\,m-10}$.

25. $\dfrac{a^2-a-56}{a^2+14\,a+49}-\dfrac{a^2-25}{a^2-2\,a-35}$.

26. $\dfrac{2\,a+2\,b}{a-b}+\dfrac{2\,a-2\,b}{a+b}+\dfrac{8\,ab}{a^2-b^2}$.

27. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4\,ab}{a^2-b^2}$.

28. $\dfrac{3\,c+5}{3\,c-5}-\dfrac{9\,c^2+25}{9\,c^2-25}$.

29. $\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}-\dfrac{x^2-4}{x^2+4}$.

30. $\dfrac{x-3}{x^2-2\,x-35}-\dfrac{x+5}{x^2-10\,x+21}+\dfrac{x-7}{x^2+2\,x-15}$.

31. $\dfrac{1}{c-2}-\dfrac{c}{c^2-4}+\dfrac{c^2}{c^3-8}$.

Express as a single fraction:

32. $4+\dfrac{x-5}{x+5}$.

33. $\dfrac{2\,a^2-9}{a^2+9}+2$.

34. $\dfrac{a^2+9}{a^2-9}-1$.

35. $2+\dfrac{2}{x^2-25}+\dfrac{2}{5\,x^2-125}$.

36. $\frac{11}{288}+\frac{17}{36}-\frac{85}{48}$.

37. $\frac{41}{204}-\frac{8}{125}+\frac{1}{51}$.

38. $\dfrac{1}{75}-\dfrac{1}{81}+\dfrac{a}{x}$.

39. $\dfrac{5\,x}{76}+\dfrac{14\,x}{95}-x$.

169. To reduce a fraction to a whole or mixed number principle 6 must be employed. We choose the denominator of the fraction as the number by which the numerator and denominator must be divided.

Ex. Reduce $\dfrac{5\,x^3-x^2+4\,x-6}{2\,x^2-3\,x+7}$ to a whole or mixed number.

Dividing both numerator and denominator by $2\,x^2-3\,x+7$,

we have
$$\frac{5}{2}x+\frac{13}{4}-\frac{15\,x+115}{4(2\,x^2-3\,x+7)}.$$

This result is a mixed number, and the first two terms are integral.

170. An *algebraic* expression is *integral* if its denominator is *numerical*. That is, if its denominator is 1, 2, 3, 4, etc.

In the result of the example in § 169, $\frac{5}{2}x$ and $\frac{13}{4}$ are integral *algebraic* expressions, though not integral *arithmetic expressions.*

EXERCISE 57

Reduce to whole or mixed numbers:

1. $\frac{4x+1}{4x-1}$.

2. $\frac{x^2+5x+6}{x+2}$.

3. $\frac{x^2+5x+6}{x-3}$.

4. $\frac{7x^2-5x+12}{3x+1}$.

5. $\frac{x^3+3x^2+3x+1}{x^2+2x+1}$.

6. $\frac{x^3+5x^2-7x+9}{x^2+3x-2}$.

7. $\frac{8x^3+64y^3}{2x+4y}$.

8. $\frac{x^4+81}{x+3}$.

9. $\frac{x+1}{x+2}+\frac{x+2}{x+3}+\frac{x+3}{x+4}$.

(Reduce each fraction separately, then add the results.)

10. $\frac{x-1}{x-2}-\frac{x-2}{x-3}$.

(Reduce to mixed numbers, then combine.)

11. $\frac{x+4}{x+5}-\frac{x+5}{x+6}$.

12. $\frac{x+2}{x+3}+\frac{x+3}{x+4}-\frac{2x+5}{x+2}$.

Multiplication

171. Multiplication of fractions involves principles 1 and 6.

Ex. 1. Multiply $\frac{18}{25}$ by 5.

$$\frac{18}{25}\cdot 5=\frac{18\cdot 5}{25}.\quad \text{(Never use cancellation.)}$$

$$\frac{18\cdot 5}{25}=5 \;\Big|\; \frac{18\cdot 5}{25}=\frac{18}{5}.$$

Ex. 2. Multiply $\frac{18}{25}$ by 3.

$$\frac{18}{25}\cdot 3=\frac{18\cdot 3}{25}=\frac{54}{25}.$$

Ex. 3. Multiply $\frac{a+2}{a^2-a-12}$ by $a+4$.

$$\frac{a+2}{(a-4)(a+3)}\cdot(a+4)=\frac{(a+2)(a+4)}{(a-4)(a+3)}=\frac{a^2+6a+8}{a^2-a-12}.$$

Ex. 4. Multiply $\frac{a+2}{a^2-a-12}$ by $a-4$.

$$\frac{a+2}{a^2-a-12}\cdot(a-4)=\frac{(a+2)(a-4)}{(a-4)(a+3)}=\frac{a+2}{a+3}.\quad \text{(Principle 6.)}$$

Ex. 5. Multiply $\frac{5}{8}$ by $\frac{4}{15}$.

$$\frac{5}{8}\cdot 4=\frac{5\cdot 4}{8}.$$

But our multiplier is $\frac{1}{15}$ of 4. Hence our product is

$$\frac{5\cdot 4}{8}\div 15 \text{ or } \frac{5\cdot 4}{8\cdot 15}=\frac{1}{6}.\quad \text{(Principles 2 and 6.)}$$

EXERCISE 58

Simplify, using factoring:

1. $\frac{17}{196}\cdot 28.$

2. $\frac{35}{289}\cdot 51.$

3. $\frac{33}{361}\cdot 76.$

4. $\frac{x-1}{x^2-4}\cdot(x-2).$

5. $\frac{a^2-5a+6}{a^2-9}\cdot(a^2-9).$

6. $\frac{m^2+6m+8}{m^2+7m+10}\cdot(m^2+5m).$

7. $\frac{15}{28}\cdot\frac{42}{25}.$

8. $\frac{(a-2)^2}{(a+2)^2}\cdot\frac{(a^2+5a+6)}{(a^2-5a+6)}.$

9. $\frac{c+5}{c-8}\cdot\frac{c^2-10c+16}{c^2-25}.$

10. $\frac{b^2+8b}{4b-1}\cdot\frac{16b^2-8b+1}{3b+24}.$

11. $\frac{18a^2b}{27ab^2}\cdot\frac{12a^3b^2}{25bc^3}\cdot\frac{15ac}{16b}.$

12. $\frac{5a^3}{7bc}\cdot\frac{98b^3}{125ac}\cdot\frac{75c^3}{28ab}.$

13. $\frac{m^2+5m+6}{m+5}\cdot\frac{m^2+7m+10}{m^2+4m+4}\cdot\frac{m+1}{m^2+4m+3}.$

14. $\frac{10c^2-21c-10}{12c^2+13c+3}\cdot\frac{3c^2-8c-3}{2c^2-c-10}\cdot\frac{4c^2+11c+6}{5c^2-13c-6}.$

15. $\left(\frac{c^2+7c+6}{c^2-3c-10}\right)\left(1-\frac{12c+17}{c^2+8c+12}\right).$

16. $\left(\dfrac{a^2-b^2}{a^2+ab-2b^2}\right)\left(1+\dfrac{4ab}{a^2-2ab+b^2}\right)$.

17. $(\frac{5}{7})(1+\frac{7}{2})$.

18. $\dfrac{x^2+xy+y^2}{x^2-81y^2}\cdot\dfrac{x+9y}{(x-y)(x^2+xy+y^2)}$.

19. $\left(1+\dfrac{x^2+x+1}{x}\right)\left(x+\dfrac{x}{x^2+2x+1}\right)$.

20. $\left(\dfrac{x+3}{x+5}+\dfrac{x+5}{x+3}\right)\left(1+\dfrac{x^2+8x+15}{x^2+6x+9}\right)$.

21. $\dfrac{x+5}{x+3}+\dfrac{x+3}{x+5}\left(1+\dfrac{x^2+8x+15}{x^2+6x+9}\right)$.

Division

172. Division depends on principles 2 and 6.

Ex. 1. $\frac{9}{5}\div 3$.

$$\frac{9}{5}\div 3=\frac{9}{5\cdot 3}=\frac{3^2}{5\cdot 3}=\frac{3}{5}.$$

Ex. 2. Divide $\dfrac{x^2-1}{x^2-4}$ by $x-1$.

$$\frac{x^2-1}{x^2-4}\div(x-1)=\frac{(x+1)(x-1)}{(x+2)(x-2)(x-1)}=\frac{x+1}{x^2-4}.$$

Ex. 3. Divide $\frac{7}{8}$ by $\frac{21}{12}$.

$$\frac{7}{8}\div 21=\frac{7}{8\cdot 21}.$$

But our divisor is $\frac{1}{12}$th of 21, hence our quotient is 12 times as large as though the divisor were 21. Hence,

$$\frac{7}{8}\div\frac{21}{12}=\left(\frac{7}{8\cdot 21}\right)12 \text{ or } \frac{7\cdot 12}{8\cdot 21}=\frac{7\cdot 2^2\cdot 3}{2^3\cdot 3\cdot 7}=\frac{1}{2}.$$

What divisor was used to produce $\dfrac{1}{2}$ from $\dfrac{7\cdot 2^2\cdot 3}{2^3\cdot 3\cdot 7}$?

Therefore, *to divide a fraction by a fraction invert the terms of the divisor and proceed as in multiplication.*

EXERCISE 59

Simplify the following:

1. $\dfrac{14}{15} \div 7.$

2. $\dfrac{102}{105} \div (-17).$

3. $-\dfrac{102}{105} \div \left(\dfrac{-17}{30}\right).$

4. $\dfrac{50\,a^2b}{33\,c^3} \div \dfrac{15\,ab^2}{44\,c^4}.$

5. $\dfrac{169(x+y)^2}{144(x-y)^3} \div \dfrac{13(x+y)}{18(x-y)^4}.$

6. $\dfrac{-288(a-b^2)}{289(a^2-b)} \div \dfrac{36(a-b^2)^2}{51(a^2-b)^2}.$

7. $\dfrac{a^2-25}{a^2-9} \div \dfrac{a^2-8\,a+15}{a^2-6\,a+9}.$

8. $\dfrac{a^2-17\,a+16}{a^2-5} \div 2(a-16).$

9. $\dfrac{a-b}{a+b} \div \dfrac{a+b}{a-b}.$

10. $\dfrac{z^3-27}{z^2+9} \div \dfrac{z^3+3\,z+9}{3\,z^3+27\,z}.$

11. $\dfrac{10\,c^2+17\,c+3}{2\,c^2+13\,c-7} \div \dfrac{2\,c^2+17\,c+21}{10\,c^2-3\,c-1}.$

12. * $\dfrac{\dfrac{x^2+2\,x-24}{x^2+10\,x+21}}{\dfrac{x^2+12\,x+36}{x^2+3\,x-28}}.$

13. $\dfrac{1+\dfrac{x^2-4\,ax+4\,a^2}{8\,ax}}{\dfrac{(x+2\,a^2)}{4\,ax}}.$

14. $\left(\dfrac{a^2+2\,ab+b^2}{4\,ab}-1\right) \div (a-b).$

15. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}.$

16. $\dfrac{\dfrac{1}{x+y}-\dfrac{1}{x-y}}{\dfrac{1}{x+y}+\dfrac{1}{x-y}}.$

17. $\dfrac{\dfrac{x-y}{y}-\dfrac{x+y}{x}}{\dfrac{x-y}{x}+\dfrac{x+y}{y}}.$

18. $\dfrac{1+\dfrac{b}{c}}{1-\dfrac{b}{c}} \div \dfrac{b+c}{b-c}.$

19. Divide $b-c$ by $c-b$.

20. $\dfrac{-x^2-y^2}{x^2+y^2} \div \dfrac{x+2\,y}{x-2\,y}.$

* *Division written in this form is called a complex fraction.*

21. $\dfrac{17^3(x-y)^3}{(115)^2(x+y)^2} \div \dfrac{17^2(x-y)^2}{(115)^2(x^2+2xy+y^2)}$.

22. $\dfrac{\dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}}{\dfrac{b}{a}-\dfrac{a}{b}} \div \dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{b^2}{a^2}-2+\dfrac{a^2}{b^2}}$.

23. Transform $\dfrac{a-b}{a+b}$ into $\dfrac{1-\dfrac{b}{a}}{1+\dfrac{b}{a}}$.

24. Transform $\dfrac{2c+x}{2c-x}$ into $\dfrac{\dfrac{2c}{x}+1}{\dfrac{2c}{x}-1}$.

Fractional Equations

173. Equations involving fractions are solved by first reducing to integral equations by means of principles of § 167, and Axiom 3, then solving as in exercise 8.

Ex. 1. Solve $\dfrac{2x+7}{x}=5-\dfrac{2}{x}$.

The l. c. m. is x (§ 167).

Multiplying both sides of the equation by x (Ax. 3),

$$x\left(\frac{2x+7}{x}\right)=x\left(5-\frac{2}{x}\right).$$

We have, $\quad 2x+7=5x-2,$

or $\quad -3x=-9,$

$$x=3. \quad \text{(Ax. 4.)}$$

Check this root by substituting in the *given* or original equation.

$$\frac{2\cdot 3+7}{3}=5-\frac{2}{3}, \quad \text{or} \quad \frac{13}{3}=5-1.$$

Ex. 2. Solve $\dfrac{x-1}{x-2}=\dfrac{x-2}{x-3}-\dfrac{1}{2}$.

$$\text{l. c. m.} =(x-2)(x-3)2.$$

Multiplying by l. c. m.

$$2x^2-8x+6=2x^2-8x+8-x^2+5x-6.$$

Transposing, $\quad x^2-5x+4=0.$

Factoring by Type III, $(x-4)(x-1)=0.$

Solving by § 161, $\quad x=4$ or 1.

Check for $x=4$. $$\frac{4-1}{4-2}=\frac{4-2}{4-3}-\frac{1}{2},$$

or $$\frac{3}{2}=\frac{2}{1}-\frac{1}{2}.$$

For $x=1$, $$\frac{1-1}{1-2}=\frac{1-2}{1-3}-\frac{1}{2}.$$

or $$0=\frac{-1}{-2}-\frac{1}{2}.$$

Ex. 3. $$\frac{1}{x-5}+\frac{1}{x+4}=\frac{3x-6}{x^2-x-20}.$$

$$\text{l. c. m.} = (x-5)(x+4).$$

Multiplying by l. c. m.

$$x+4+x-5=3x-6.$$

Transposing, $$-x=-5.$$

$$x=5.$$

Check: $$\frac{1}{5-5}+\frac{1}{5+4}=\frac{3\cdot5-6}{5^2-5-20}.$$

or $$\frac{1}{0}+\frac{1}{9}=\frac{9}{0}.$$

But it is not allowable to divide by zero.

Hence 5 is not a root. The equation has *no* solution. It is not an equation of condition. It is simply a statement that two numbers are equal, § 14, but the statement is not true.

EXERCISE 60

Solve and verify the following:

1. $\dfrac{2x}{3}=8.$

2. $\dfrac{3}{2x}=8.$

3. $5=\dfrac{15}{x}.$

4. $6=\dfrac{11}{x}.$

5. $\dfrac{5x}{2}+1=3x+\dfrac{1}{2}.$

6. $\dfrac{1}{x-1}+\dfrac{1}{x-2}=\dfrac{3x+2}{x^2-3x+2}.$

7. $\dfrac{1}{x-1}+\dfrac{1}{x-2}=\dfrac{5x-9}{x^2-3x+2}.$

8. $\dfrac{2x}{5}+\dfrac{3x+4}{2x-1}=\dfrac{6x-15}{15}.$

(Unite the first and third fractions before clearing the equation of fractions.)

9. $\frac{2x}{5}+\frac{3x+4}{2x-1}=\frac{6x+5}{15}.$

10. $\frac{x}{2}+\frac{5a}{3}=4x-2a.$ Solve for x.

11. $\frac{1}{x}+\frac{1}{a}=\frac{5}{x}+\frac{2}{a}.$ Solve for x.

12. $\frac{4y-1}{6}-\frac{2y+7}{4y-8}=\frac{8y+7}{12}.$

13. $\frac{t+a}{t-a}=\frac{3}{4}.$ Solve for t.

14. $\frac{t+a}{t-a}-\frac{t-a}{t+a}=\frac{t-3a}{t^2-a^2}.$

15. $\frac{6x^2+14x+8}{3}=\frac{4(3x-5)(x+4)}{6}+x.$

16. $\frac{x}{a+b}-\frac{x}{a-b}=\frac{a-b}{(a+b)^2}.$

17. $\frac{15z^2+5}{9z^2-25}=\frac{2z}{5+3z}-\frac{3z}{5-3z}.$

18. $\frac{3m^2-2m+1}{m+3}-\frac{3m^2+m-2}{m-3}=\frac{-19m^2+3m+1}{m^2-9}.$

19. $\frac{m+a}{m-a}+\frac{2m-a}{2m+a}=2.$ Solve for m.

20. $\frac{4z+1}{3}-\frac{2z+3}{5}-\frac{3z-5}{10}=\frac{12z-4}{15}.$

21. $\frac{3k+a}{2a}-\frac{5k-a}{4a}+\frac{2k+3a}{6a}=\frac{7k+2a}{10a}.$ Solve for k.

22. $\frac{5z+2d^2}{7d^2}-\frac{2z+8d^2}{z+9d^2}=\frac{15z-d^2}{21d^2}.$ Solve for z.

23. $\frac{1}{y-5}+\frac{1}{y-6}=\frac{5y-26}{y^2-11y+30}.$

24. $\dfrac{1}{m-5}+\dfrac{1}{m-6}=\dfrac{5m-23}{m^2-11m+30}$.

25. $\dfrac{6k}{k+5}-\dfrac{2k}{2-k}=\dfrac{8(k^2-4)}{k^2+3k-10}$.

26. $\dfrac{1}{z-2}-\dfrac{1}{z-4}=\dfrac{1}{z-3}-\dfrac{1}{z-5}$.

(Combine the first two fractions, then the second two fractions before clearing the equation of fractions.)

27. $\dfrac{cy+my}{c}-y=\dfrac{1}{c-m}+\dfrac{1}{m}$.

28. $\dfrac{d-3}{d-4}-\dfrac{d-4}{d-5}=\dfrac{d-1}{d-2}-\dfrac{d-2}{d-3}$.

29. $\dfrac{x+1}{x+3}+\dfrac{x-2}{x-4}=\dfrac{2x+8}{x+4}$.

30. The denominator of a fraction is 5 more than the numerator. If 9 is added to the numerator and 7 subtracted from the denominator, the result is $1\frac{1}{23}$. Find the fraction.

31. Find three consecutive numbers which will satisfy these conditions: if the smallest is multiplied by six and three subtracted from the product, this product divided by the greatest number will give a quotient of 5.

32. I have two proper fractions. The denominator of each is one more than its numerator, the numerator of the first, the numerator of the second, and the denominator of the second are consecutive numbers, and when the greater fraction is subtracted from the lesser the quotient is $-\frac{1}{42}$. Find the fractions.

33. Divide 48 into two parts, such that the fraction formed by these parts is $\frac{5}{7}$.

34. Two Ohio cities are 120 miles apart. Two trains running between these cities have a difference in rate of 5 miles per hour, and the difference in time it takes them to make the run is 20 minutes. Find the rate of each train.

35. A local train loses six of its passengers at the first stop, one third of the remainder at the second stop, one half of the remainder at the third stop, and the 30 who then remain ride to the end of the line. How many passengers on the train when it started?

36. In example 35, the first station is $\frac{1}{3}$ as far out as the second, the third is 36 miles beyond the second, the fourth 49 miles beyond the third, and the length of the run is 117 miles. At 2¢ per mile per passenger, how much did the railway company receive?

37. A number exceeds the sum of its one third, one fourth, and one fifth by 13. What is the number?

38. An oil tank can be filled by one pump in 8 hours, and by a second pump in 14 hours. How long does it take to fill the tank when both pumps are working?

Let $x =$ the time when both pumps are working.

Then $\frac{1}{x}$ is the amount of work both pumps do in 1 hour.

$\frac{1}{8}$ is the amount of work the first pump does in 1 hour, and

$\frac{1}{14}$ is the amount the second pump does in 1 hour.

Then $\frac{1}{8}+\frac{1}{14}=\frac{1}{x}$. *Solve.*

39. An oil tank 20′ in diameter can be filled by one pump in $177\frac{1}{3}$ hours, and by a second pump in 64 hours. How long does it take to fill the tank when both pumps are working?

40. In example 39, the cylinder of the larger pump is 10″ in diameter, the stroke of the piston is 12″, and there are 15 strokes of the pump per minute. What is the capacity of the tank in cubic feet? In gallons?

41. Separate 45 into two such parts that one part divided by the other will give a quotient of 2 and a remainder of 9.

42. Separate 45 into two such parts that one part divided by the other will give a quotient of 5 and a remainder of 37.

43. Separate 45 into two such parts that one part divided by the other will give a quotient of 5 and a remainder of 48.

44. Two men, 58 miles apart, start at the same time to travel toward each other. The first travels 7 miles in 2 hours, the second travels 15 miles in 4 hours. How far does the first one travel before they meet?

45. The denominator of a fraction is 5 less than the numerator. If 5 is added to the numerator the value of the fraction is then $\frac{7}{2}$. Find the fraction.

46. Two men, A and B, 58 miles apart, start at the same time to travel toward each other. A travels 7 miles in 2 hours and B travels 15 miles in 4 hours. A meets with an accident and is delayed 20 minutes. How far does B travel before they meet?

47. Separate a into two parts such that one part divided by the other will give a quotient of g and a remainder of c.

48. Separate 42 into three parts such that the second shall be four times the first and the third 4 times the second.

49. Separate c into three parts such that the second shall be m times the first and the third m times the second.

50. A has $1200 out at interest, part at 6 % and the balance at 5 %. The part at 6 % brings as much interest in 4 years as the part at 5 % brings in 6 years. What is his total amount of interest per year?

51. $\dfrac{4(58-x)}{15}=\dfrac{2x}{7}.$

Multiplying by 105,

$$28 \cdot 58 - 28x = 30x.$$
$$58x = 28 \cdot 58.$$
$$x = 28.$$

(Indicate such products and save computation.)

52. $\dfrac{24-x}{16}=9+\dfrac{5x}{48}.$ **53.** $\dfrac{5x-2}{6}+\dfrac{7x-6}{5x+13}=\dfrac{15x-5}{18}.$

Supplemental Applied Mathematics

1. The specific gravity of ice (ratio of the weight of a cubic foot of ice to that of a cubic foot of water) is .92. How much water is there in a cake of ice $3' \times 10.5'' \times 10.5''$? What will the ice weigh?

2. What is the third dimension of a cake of ice $1' \times 1'$, weighing 50 pounds?

3. A piece of iron $4'' \times 8'' \times 1'$ is placed in a tank of water. How many liters of water did it displace?

4. One cubic foot of steel immersed in water weighs how much?

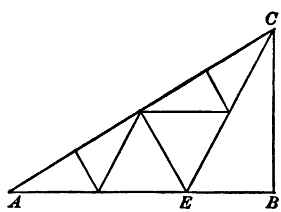
5. The specific gravity of sea water is 1.025, and that of ice is .92. Find the difference in weight between one cubic foot of sea water and one cubic foot of ice.

6. A piece of ice $3' \times 1' \times 1'$ is dropped overboard from an ocean liner. How much of the ice is submerged in the ocean?

7. A tank of water $18' \times 8'$ and $6'$ deep is frozen to a depth of $7''$. Find the value of the ice at 42¢ per hundred pounds?

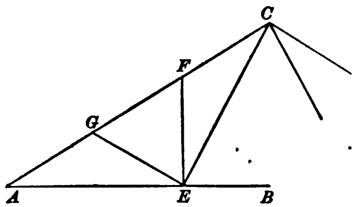
8. Air is 14.43 times as heavy as hydrogen gas. 8500 cubic feet of hydrogen have been pumped into a balloon. What weight will it lift?

9. In testing 100 pounds of steam coal there was found 8.3 pounds of ash, .932 pounds sulphur. What was the per cent of carbon?



10. The span of a roof is $42'$. The pitch of AC is 30°. The member* CE bisects the angle ACB. Find the lengths of all the members used.

* The pieces used in forming a roof truss are called "members," "angle irons," or "angles."



11. Find the lengths of the angles if the roof in example 11 is of the form of the truss shown in the accompanying figure. (EFG is equilateral.)

12. A $1\frac{1}{8}''$ rainfall on 20 acres of land is how many barrels of water?

13. A school building has eight drinking fountains, each flowing $1\frac{1}{2}$ gallons per minute. These are supplied by a pump whose cylinder is 4″ in diameter and 10″ long. How many strokes per minute must the pump make to keep these fountains running?

14. The illumination given by any light is inversely as the square of the distance $\frac{I_1}{I_2} = \frac{d_2^2}{d_1^2}$.

A light 6 feet from a table is moved 3 feet from the table. How much more light does the table now receive?

15. A 16 candle power and a 4 candle power electric light are placed on opposite sides of a screen. The 4 candle power is 3′ from the screen. At what distance must the 16 candle power be placed that each side of the screen may receive the same amount of light?

16. A chandelier directly over a table contains four 16 candle power carbon filament lights. These lights are 4′–6″ from the ceiling. A new chandelier fitted with four tungsten lights 3′–6″ from the ceiling is put in. These tungsten lights give $1\frac{1}{2}$ times as much light as the old 16 candle power. Does the table receive more or less light and how much?

17. A 16 candle power electric light is 5′ above a table and does not give sufficient light. To remedy this defect a wire is attached to the socket and brought down to a reading lamp containing a tungsten burner which is 18″ above the table. How much more light does the table receive?

18. In cooling iron shrinks about $\frac{1}{8}''$ per lineal foot after it is cast. A casting must be $2'$–$1''$ in length and $1'$–$8''$ square after cooling. What were its dimensions when cast?

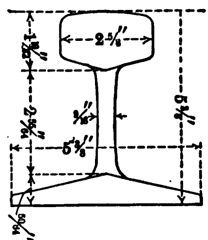
19. A gas engine cylinder is to be $4''$ in diameter and $4''$ long. What is its size before cooling?

20. A gas engine cylinder is to be $3\frac{1}{2}''$ in diameter and $4''$ long. Find dimensions of the pattern from which it is cast.

21. What size bolt would you use in a $0.344''$ hole?

22. What size bolt would you use in a $1.491''$ hole?

23. A man was standing behind a target during target practice. Those doing the firing were 1 mile away and the velocity of their projectiles was $1150'$ per second. Did the projectile strike the target before the sound reached there? What was the difference between the time the sound of firing and the projectile reached the target?



24. Estimate the weight per lineal foot of the Steel Tee Rail in the accompanying figure. Disregard the round corners; consider them as angles.

25. If a plumber needs to change the direction of a pipe by $45°$, he calls the hypotenuse AC of the triangle ABC equal to $BC + \frac{5}{12} BC$. What is the error when $BC = 30''$? Which is easier to compute, the steam fitter's method or the correct method?

26. In estimating material for a bias ruffle a dressmaker calls the length of the bias $\frac{4}{3}$ the width of the goods. What is the error when the goods is $27''$ wide?

27. A plumber makes a $45°$ turn across a hallway $10'$ wide. The hall is $46'$ long and is piped the entire length. What length of pipe is used?

28. In example 27, how many feet of pipe would be nee if a 30° angle were used in making the cross over?

29. In a triangle ABC, right-angled at B, $\angle A$ is 30°. such triangles many mechanics estimate that AC is $1\frac{1}{2}$ ti AB. How nearly correct is this when AB is 10′? Is method as easy to compute as the correct one, namely,

Let $2x = AC$, then $x = BC$, and $AB = \sqrt{(2x)^2 - (x)}$

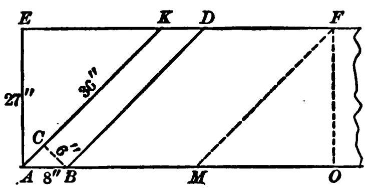
Let $AB = a$, then $\sqrt{(2x)^2 - x^2} = a$,

$$x = \frac{a}{3}\sqrt{3}.$$

30. A house is 26 feet wide. The pitch of the roof is Find the length of the rafters, no allowance being made extensions at the eaves.

31. The width of a house is 28′ and the roof has a pitc 30°. What length rafters must the carpenter cut if the raf project 14″ beyond the house at the eaves?

32. Use dressmaker's rule in finding the number of yard goods necessary to make a 6-inch bias ruffle for a skirt 4 ya around. The goods is wide and no strip of ruff is to be less than 35″ in len Let AF be the piece of go then $AK = 36″$, the lengt each strip of ruffling. It therefore require 6 strip ruffling for this 4-yard skirt. The cut AB along the selv is 8″. The amount of goods required is therefore $6 \times 8″$] the 27″ (width of goods MO) wasted at the corner, or $2\frac{1}{8}$ ya



33. Find the cost of two 3″ bias ruffles made from go 30″ wide and costing 95¢ per yard, the skirt being 4 ya around.

34. Silk may be purchased "on the bias." This avoids] *ing for the* waste at the corner. Find the cost of three $1\frac{1}{2}″$

ruffles to be used on a 6-yard silk skirt, silk 21″ wide and costing $1.10 per yard.

35. The braces for a billboard are to have an angle of 60° with the ground. What length must the braces be cut if the foot of the brace is 8′ from the board?

36. A piece of steel is $1\frac{1}{2}''$ by 5″. It weighs 225 pounds. Find its length.

37. The volume of a cube is .512 cubic foot. Find its surface.

38. The water in an irrigating ditch flows $3\frac{1}{2}$ miles per hour. It must supply a 160-acre farm with 1″ of water per week. What is the area of a cross section of the ditch?

39. Water is flowing through an 8″ water main at the rate of 10′ per second. How many barrels of water will it deliver per hour?

40. A farmer had a pond of 6 acres which was frozen to a depth of 10″. He sold the ice to a dealer at 12¢ per hundred pounds. How much did he receive?

41. One diagonal of a rhombus and a side of the rhombus are each 18″. Find the other diagonal, the angles, and the area.

42. A bar of copper $4'' \times 6'' \times 3'$ is drawn into a $\frac{1}{8}''$ wire. How long is the wire? What does the coil weigh?

43. The area of a cold-air box is to be $\frac{1}{6}$ less than the combined areas of the hot-air pipes. One hot-air pipe is 10″ in diameter, one is 8″, and the remaining six are each 6″. Find the area of the cold air box.

44. From a town C one train goes north at 32 miles an hour. One hour later a second train goes east at 30 miles an hour. How far apart are they 3 hours after the first train started?

45. From a town C a train goes north at 32 miles an hour. 5 hours later a second train follows the first at 40 miles an hour. How far apart are they 8 hours after the second train started?

46. Chocolate contains 12.9 % protein; cocoa, 21.6 %. How much chocolate will furnish as much protein as $\frac{1}{2}$ pound cocoa?

47. Chocolate contains 48.7 % fat, and cocoa, 28.9 %. One half pound cocoa measures 2 cups. How many cups cocoa will furnish as much fat as $\frac{1}{2}$ pound chocolate?

48. As to the quantity of fat, which is cheaper to use, cocoa or chocolate, if 8 ounces chocolate cost 22¢ and $\frac{1}{2}$ pound cocoa costs 25¢?

49. If halibut costs 18¢ per pound and 17.7 % is refuse, find cost per pound of edible portion.

50. Haddock costs 12¢ per pound; 51 % is refuse. Which is cheaper fish, halibut or haddock?

51. Boned and dried codfish sells for 16¢ per pound; dried codfish for 10¢ per pound. From the latter there is a loss of 20 %. Which is cheaper?

52. Whitefish contains 12.8 % protein; 43.6 % is refuse. It sells for 16¢ per pound. Porterhouse steak contains 19.1 % protein; 12.7 % is refuse. It sells for 30¢ per pound. Which food contains more protein for less money?

53. Bass contains 9.3 % protein; 54.8 % is refuse. It sells for 12¢ per pound. Which kind of fish is cheaper, whitefish or bass?

54. Herring contains 11.2 % protein; 42.6 % is refuse; it sell for 10¢ per pound. Perch contains 7.3 % protein; 62.5 % is waste; it sells for 10¢ per pound. Which fish contains more protein for less money?

55. Pike contains 7.9 % protein; 57.3 % is refuse; it sells for 12¢ per pound. Round steak contains 27.6 % protein; it sells for 16¢ per pound. Which is the cheaper food?

56. Canned salmon sells for 18¢ per can; it weighs $1\frac{1}{4}$ pounds; contains 19.5 % protein, 14.2 % waste. Sardines sell for 25¢ per can, which weighs $11\frac{3}{4}$ ounces; they contain 23.7 % protein *and 5 % waste.* Which is cheaper to use?

57. Dried beef contains 39.2% protein. It costs 30¢ per pound. Which is cheaper, dried beef or salmon?

58. Express graphically the edible quantities of haddock, halibut, whitefish, bass, herring, perch, pike, and canned salmon that can be purchased for 25¢.

59. Express graphically the quantities of protein per pound in these fish.

CHAPTER X

Proportion

174. Review § 173 and exercise 60. The relation of one number to another is often expressed in fractional form. These fractions are known as **ratios**. Thus, the ratio of 2 to 3 is written $\frac{2}{3}$. This was first written $2 \div 3$, then the division sign was modified to $2:3$, now the fractional form is considered best.

A **proportion** is the equality of two ratios. It is therefore simply a fractional equation of two terms.

Thus, $\frac{a}{b}=\frac{c}{d}$ is a proportion. This is read a divided by b is equal to c divided by d.

The numerators are the **antecedents**.

The denominators are the **consequents**.

The first antecedent and the last consequent are the **extremes**.

The first consequent and the second antecedent are the **means**.

Thus, a and d are extremes and b and c the means.

Properties of Proportion

1. *The product of the extremes is equal to the product of the means.*

This is easily proved by clearing the equation (proportion),

$$\frac{a}{b}=\frac{c}{d},$$

of fractions.

Whence, $$ad = bc.$$

2. *If the product of two numbers is equal to the product of two other numbers, a proportion may be formed, making one product the means and the other product the extremes.*

Thus, $$xy = mc.$$

Dividing by $y \cdot m$, $$\frac{x}{m} = \frac{c}{y}.$$

Had you divided by $y \cdot c$, the proportion would have been

$$\frac{x}{c} = \frac{m}{y}. \quad \text{(See 6, § 174.)}$$

3. *If four quantities are in proportion, they are in proportion by composition.*

Let $$\frac{a}{b} = \frac{c}{d}.$$

Then, $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

4. *If four quantities are in proportion, they are in proportion by division.*

Let $$\frac{a}{b} = \frac{c}{d}.$$

Then, $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

5. *If four quantities are in proportion, they are in proportion by composition and division.*

Let $$\frac{a}{b} = \frac{c}{d}. \tag{1}$$

By 3, $$\frac{a+b}{b} = \frac{c+d}{d}. \tag{2}$$

By 4, $$\frac{a-b}{b} = \frac{c-d}{d}. \tag{3}$$

Dividing (2) by (3), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \tag{4}$$

6. *If four quantities are in proportion, they are in proportion by alternation.*

Thus, if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$.

175. In a *mean proportion* the means are equal.

Thus, $\frac{a}{x} = \frac{x}{d}$ is a mean proportion.

Solving $x = \sqrt{ad}$.

That is, a mean proportional between a and d is the square root of their product.

The last consequent of a mean proportion is a *third proportional* to the other two numbers.

Thus in $\frac{a}{x} = \frac{x}{d}$, d is a third proportional to a and x.

The *fourth proportional* is the last consequent in such a proportion as $\frac{a}{b} = \frac{c}{d}$, where no two terms are alike.

EXERCISE 61

1. Find a mean proportional between 4 and 9.
2. Find a mean proportional between 16 and 25.
3. Find a mean proportional between 289 and 256.

$$\frac{289}{x} = \frac{x}{256},$$

or, $x = \sqrt{289 \cdot 256}$. But $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$

Then, $x = \sqrt{289} \cdot \sqrt{256}$

$$= 17 \cdot 16 = 272.$$

4. Find the mean proportional between 121 and 729.
5. Two lines are 196′ and 25′, respectively. Find a line equal to their mean proportional.
6. What is the mean proportional between 1.44″ and 2.56″.
7. *Find a fourth* proportional to 2′, 3′, 8′.

8. Find a fourth proportional to 5, 9, 15.

9. Find a fourth proportional to 6, 9, 3.

10. Find a fourth proportional to 5, $7\frac{1}{2}$, 9.

Note that in such proportions as $\frac{2}{3} = \frac{6}{x}$, x may be obtained by inspection. For since $6 = 3 \cdot 2$ (the first numerator), x must be $3 \cdot 3$ (the first denominator). Or, reading vertically instead of horizontally, since the first consequent is $1\frac{1}{2}$ times the first antecedent, the second consequent must be $1\frac{1}{2}$ times the second antecedent.

11. Find a third proportional to 2, 8.

12. Find a third proportional to 5, 15.

13. Find a third proportional to 5, 9.

14. Find a mean proportional between .0529 and 529.

15. Find a mean proportional between $\frac{x^2 + 5x + 6}{x + 5}$ and $\frac{x^2 + 8x + 15}{x + 2}$.

16. Find a mean proportional between $\frac{2x + 5}{x^2 + 8x - 7}$ and $\frac{x + 7}{2x^2 + 3x - 5}$.

17. Find a fourth proportional to $x^2 + 9x + 20$, $x^2 - 3x - 28$, and $2x^2 + 19x + 45$.

18. In a semicircle, if a perpendicular is dropped to the diameter, from a point in the circumference, the perpendicular is a mean proportional between the segments of the diameter. The diameter of the circle is 20 and the perpendicular to it from the circumference is 8. Find the segments of the diameter.

19. The segments AB and BC of a diameter AC are 4″ and 9″ respectively, find the length of the perpendicular to AC erected at B and extending to the circumference.

20. $\frac{2x+8}{2x-3} = \frac{5x+11}{5x-11}$. Solve, using § 174, 5, before clearing of fractions.

21. $$\frac{(x+7)+(3y-1)}{(x+7)-(3y-1)}=\frac{7}{2}.$$
$$5x-7y=-9.$$

(Use § 174, 5, on the first equation before clearing it of fractions.)

Ratio plays a very important part in science, though the ratio idea is often disguised to such an extent by the scientific notation that the pupil thinks in other terms than those of ratio or measurement.

For example, the mysteries of Specific Gravity disappear when one *feels* that the specific gravity is simply the ratio of a volume of some substance to an equal volume of some substance taken as a standard.

The standard for liquids and solids is water.

One cubic centimeter (c.c.) of water weighs 1 gram, or 1 cu. ft. weighs $62\frac{1}{2}$ pounds.

For gases the standard is usually hydrogen; sometimes air, which is 14.44 times as heavy as hydrogen, is used.

Ex. A cubic foot of steel weighs 490 lb. Find specific gravity of steel.

Specific gravity of steel $= \frac{490}{62.5} = 7.84.$

It is customary to write specific gravity in a decimal form, not as a common fraction.

Units to be remembered:

$1''$ = 2.54 centimeters.
1 liter = 1000 cubic centimeters (c.c.).
1 kilogram = 1000 grams.
1 c.c. water weighs 1 gram.
1 liter hydrogen weighs 0.09 gram.
Specific gravity air (hydrogen standard) is 14.44.

22. Ice weighs 57.5 pounds to the cubic foot. Find its specific gravity.

23. The specific gravity of oak is 0.8. Find the weight of 1 cubic foot.

24. A cubic foot of lead weighs 706 pounds. Find its specific gravity.

25. A cubic foot of copper weighs 550 pounds. Find its specific gravity.

26. The specific gravity of aluminum is 2.6. Steel is how many times as heavy?

LIST OF CONSTRUCTIONS

LIST OF THEOREMS

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